CHIRAL ALGEBRAS

Alexander Beilinson
Vladimir Drinfeld
The University of Chicago
Department of Mathematics
5734 S. University Ave
Chicago, IL 60637

Contents

Introduction

Chapter One: Axiomatic Patterns

§1 Pseudo-tensor categories

§2 Complements

§3 Compound tensor categories

§4 Rudiments of compound geometry

Chapter Two: Geometry of $\mathcal{D}_X$-schemes

§1 $\mathcal{D}$-modules: recollections and notation

§2 The compound tensor structure

§3 $\mathcal{D}_X$-schemes

§4 The spaces of horizontal sections

§5 Lie* algebras and algebroids

§6 The Tate extension

§7 Coisson algebras

§8 Other settings

Chapter Three: Chiral Basics

§1 Chiral operations

§2 Relation to "classical" operations

§3 Chiral algebras and modules

§4 Local associative algebras

§5 Chiral algebras as factorization algebras

§6 Operator product expansions

§7 Chiral enveloping algebras

§8 BRST, alias semi-infinite, homology

§9 Chiral differential operators

§10 Other settings

§11 Chiral homology

§12 Utensils

References

INTRODUCTION

В настоящей работе не содержатся сведения, которые могли бы составить предмет изобретения или открытия

Анон., "Акт Экспертизы", 1956 (?)[+]

0.0. This work is an exposition of basic chiral algebra theory. Chiral algebras take their origin in mathematical physics; they lie at the heart of conformal field theory. Mathematicians usually look at them through the formalism of vertex operator algebras (see, e.g., [Bor], [FLM], [FHL], [DL], [LZ], [K]), which incorporates chiral algebras in representation theory of infinite-dimensional algebras. We follow a different approach which tastes more of algebraic geometry than of representation theory.

0.1. Chiral algebras are "quantum" objects. Let us describe first the corresponding "classical" objects; we call them *coisson* algebras ("coisson" may be considered as an abbreviation for "chiral Poisson" or "compound Poisson", the word "compound" being related with the notion of compound tensor category which we discuss in 0.4). In fact "coisson algebra" is a new name for a well known class of objects. Informally a coisson algebra is defined by a local Poisson bracket on a space of classical fields. In the simplest case where

[+] This article does not contain any information that could be the subject of an invention or a discovery. Anon., "Expert Certification", 1956 (?)

"classical field" means "a function $u(x)$" a mathematical physicist would define a local Poisson bracket by a formula like

$$\{u(x), u(y)\} = \sum_{i=1}^{n} \varphi_i(x, u(x), u'(x), \dots)\delta^{(i)}(x-y), \quad \varphi_i \in A. \tag{0.1.1}$$

Here A denots the algebra of functions of $x, u, u', \dots, u^{(k)}$ (k is not fixed). Of course in the l.h.s. of (0.1.1) $u(x)$ is understood as a functional $u \mapsto u(x)$ on the space of classical fields. The bracket should be skew-symmetric and satisfy the Jacobi identity, so the collection of functions φ_i should satisfy certain differential equations. For $f \in A$ and every x we have the functional $\ell_{f,x}$ on the space of classical fields defined by $\ell_{f,x}(u) = f(x, u(x), u'(x), \dots)$. It follows from (0.1.1) that

$$\{\ell_{f,x}, \ell_{g,y}\}(u) = \sum_i \varphi_i^{f,g}(x, u(x), u'(x), \dots)\delta^{(i)}(x-y) \tag{0.1.2}$$

for some $\varphi_i^{f,g} \in A$. Formula (0.1.2) has a more symmetric version:

$$\{\ell_{f,x}, \ell_{g,y}\}(u) = \sum_{i,j} \partial_x^i \partial_y^i \{\psi_{ij}^{f,g}(x, u(x), u'(x), \dots)\delta(x-y)\}, \quad \psi_{ij}^{f,g} \in A \tag{0.1.3}$$

Of course $\psi_{ij}^{f,g}$ are not uniquely determined by $\varphi_i^{f,g}$; we must take into account the relations

$$(\partial_x + \partial_y)(a\delta) = (\partial_x a) \cdot \delta, \quad a \in A, \tag{0.1.4}$$

where $\delta = \delta(x-y)$.

Now from the algebraic point of view the primary object is the algebra A rather than the "space of classical fields". For instance one can take $A = \mathbb{C}[x, u, u', u'', \dots]$. More generally, A can be any commutative differential unital algebra over $\mathbb{C}[x]$, and a "classical field" is a homomorphism of differential unital $\mathbb{C}[x]$-algebras $A \to B$ where B is, e.g., the algebra of C^∞-functions of $x \in \mathbb{R}$. A choice of a set of generators and defining relations for the differential $\mathbb{C}[x]$-algebra A identifies classical fields with collections of functions satisfying certain systems of differential equations.

Let us try to formulate in terms of A what is a "local Poisson bracket on the space of classical fields". A glance at (0.1.4) and the r.h.s. of (0.1.3) shows that such an object is defined by a mapping

$$A \otimes_{\mathbb{C}} A \to V \tag{0.1.5}$$

where V is the module over $\mathbb{C}[\partial_x, \partial_y]$ generated by the symbols $a\delta$, $a \in A$, with the defining relations (0.1.4). We prefer another interpretation of V: V is the module over $\mathbb{C}[x, y, \partial_x, \partial_y]$ generated by the symbols $a\delta$, $a \in A$, with the defining relations (0.1.4) and also the following ones:

$$f(x,y) \cdot (a\delta) = (f(x,x)a) \cdot \delta, \quad f \in \mathbb{C}[x,y],\ a \in A. \tag{0.1.6}$$

The mapping (0.1.5) should satsify certain properties. First of all it should be a morphism of $\mathbb{C}[x, y, \partial_x, \partial_y]$-modules (notice that $A \otimes A$ is a module over $\mathbb{C}\left[x, \frac{\partial}{\partial_x}\right] \otimes \mathbb{C}[x, \partial_y] = \mathbb{C}[x, y, \partial_x, \partial_y]$). Before discussing the other properties let us rewrite (0.1.5) in geometrical terms. An algebra A over $\mathbb{C}[x]$ is the same as a quasicoherent $\mathcal{O}_X$-algebra $\mathcal{A}$ where $X := \operatorname{Spec} \mathbb{C}[x]$ is the affine line. Differential $\mathbb{C}[x]$-algebras correspond to $\mathcal{D}_X$-*algebras* (see 0.2). The $\mathcal{D}_{X \times X}$-module corresponding to the $\mathbb{C}[x, y, \partial_x, \partial_y]$-module V from (0.1.5) is canonically isomorphic to $\Delta_* \mathcal{A}$ where $\Delta: X \to X \times X$ is the diagonal embedding. So (0.1.5) is equivalent to a $\mathcal{D}_{X \times X}$-module morphism

$$\mathcal{A} \boxtimes \mathcal{A} \to \Delta_* \mathcal{A}. \tag{0.1.7}$$

We denote by $\boxtimes$ the external tensor product, i.e., $\mathcal{A} \boxtimes \mathcal{A} := p_1^* \mathcal{A} \otimes p_2^* \mathcal{A}$ where p_1 and p_2 are the projections $X \times X \to X$.

Notice that (0.1.7) makes sense for an arbitrary smooth variety X. In 0.6 we will define a coisson algebra on X as a commutative (= commutative associative unital) $\mathcal{D}_X$-algebra $\mathcal{A}$ with a $\mathcal{D}$-module morphism (0.1.7) satisfying certain properties which actually mean that (0.1.3) is a Poisson bracket on the "space(s) of classical fields." To formualte these properties in a natural way we need some polylinear algebra in the category of $\mathcal{D}_X$-modules (see 0.2 – 0.5).

Remarks. (i) The notions of Poisson $\mathcal{O}_X$-algebra or Poisson $\mathcal{D}_X$-algebra are inadequate for expressing the idea of "Poisson bracket on the space(s) of classical fields": a Poisson bracket on $\mathcal{A}$ is a morphism $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A}$ while (0.1.7) is an object of different nature. The reason is that $\operatorname{Spec} \mathcal{A}$ is the space of "jets of classical fields", not the space of classical fields.

(ii) The reader can compare our notion of coisson algebra with other ways of formalizing the notion of "Poisson bracket on the space of classical fields" (see [DG1], [DG2], [Ku], [KuM], [Ma] and references therein). As far as we understand, our approach is essentially equivalent to that of [KuM] and [Ma].

0.2. Let X be a smooth complex algebraic variety. Consider the category $\mathcal{M}^\ell(X)$ of left $\mathcal{D}$-modules on X (see e.g. [B]). The $\mathcal{O}_X$-tensor product $\otimes$ makes it a tensor category with unit object $\mathcal{O}_X$. Commutative algebras in $\mathcal{M}^\ell(X)$ (*commutative $\mathcal{D}_X$-algebras*) are just quasicoherent commutative $\mathcal{O}_X$-algebras equipped with a flat connection along X. We call an X-scheme equipped with a flat connection along X a *$\mathcal{D}_X$-scheme*; so the spectra of commutative $\mathcal{D}_X$-algebras are $\mathcal{D}_X$-schemes affine over X.

A standard example: for any X-scheme $Y \to X$ the space $\mathcal{J}(Y/X)$ of ∞-jets of sections of Y/X is a $\mathcal{D}_X$-scheme. Horizontal sections of $\mathcal{J}(Y/X)$ are the same as arbitrary sections of Y/X. One may view the (closed) $\mathcal{D}_X$-subschemes of $\mathcal{J}(Y/X)$ as systems of differential equations on sections of Y/X (see [G]).

What makes the tensor category $\mathcal{M}^\ell(X)$ substantially different from, say, the tensor category of $\mathcal{O}_X$-modules is the absence of duals. Precisely, for $F \in \mathcal{M}^\ell(X)$ the dual object in the tensor category sense (see e.g. [D]) exists iff F is coherent as an $\mathcal{O}_X$-module (i.e., M is a vector bundle with an integrable connection). For example, for a group $\mathcal{D}_X$-scheme G one has its Lie coalgebra $CoLie(G)$ but we cannot dualize it to get the Lie algebra (unless G is finite dimensional as a usual scheme). We will see that the category $\mathcal{M}^\ell(X)$ carries a richer structure (that of *compound tensor category*) which remedies the above flaw.

0.3. Consider the category $\mathcal{M}(X)$ of right $\mathcal{D}$-modules on X. Let I be a finite non-empty set; denote by $\Delta^{(I)}: X \hookrightarrow X^I$ the diagonal embedding. For an I-family of $\mathcal{D}$-modules

$L_i \in \mathcal{M}(X)$ and $M \in \mathcal{M}(X)$ set

$$P_I^*(\{L_i\}, M) := \mathrm{Hom}(\boxtimes L_i, \Delta_*^{(I)} M) \tag{0.3.1}$$

Elements of $P_I^*(\{L_i\}, M)$ are called $*$ *I-operations*; they are X-local (since $\Delta_*^{(I)} M$ sits on the diagonal). The $*$ operations compose in a natural way[1], just as polylinear maps between vector spaces do (i.e., for a surjective map $\pi: J \to I$, a J-family $\{K_j\}$ of $\mathcal{D}$-modules, and $\varphi \in P_I^*(\{L_i\}, M)$, $\psi_i \in P^*_{\pi^{-1}(i)}(\{K_j\}, L_i)$ we have $\varphi(\psi_i) \in P_J^*(\{K_j\}, M)$). The composition is associative; if $|I| = 1$ then $P_I^* = \mathrm{Hom}$. We call such data of operations (or "polylinear maps") between the objects of a category a *pseudo-tensor structure*. Thus $\mathcal{M}(X)$ carries a canonical pseudo-tensor structure; $\mathcal{M}(X)$ equipped with this structure is denoted by $\mathcal{M}(X)^*$.

One may view pseudo-tensor categories as a straightforward generalization of operads (see e.g. [L]): an operad is just a pseudo-tensor category with single object.

The notions of algebras, modules over them, etc., make perfect sense in any pseudo-tensor category. For example, a Lie algebra in $\mathcal{M}(X)^*$ (or simply a *Lie* algebra on* X) is a $\mathcal{D}$-module L together with a binary $*$ operation $[\ \ ] \in P_2^*(\{L, L\}, L)$ which is skew-symmetric and satisfies Jacobi identity.

For $M \in \mathcal{M}(X)$ set $h(M) := M \underset{\mathcal{D}_X}{\otimes} \mathcal{O}_X$ (the sheaf of middle de Rham cohomology). There is a canonical map $P_I^*(\{L_i\}, M) \to \mathrm{Hom}(\otimes h(L_i), h(M))$ compatible with composition of operations. Therefore h sends Lie* algebras to the sheaves of usual Lie algebras. In fact, many important Lie algebras (including the Virasoro and affine Kac-Moody algebras) arise from Lie* algebras.

0.4. Let us identify the categories of left and right $\mathcal{D}$-modules by the usual equivalence $\mathcal{M}(X) \underset{\sim}{\to} \mathcal{M}^\ell(X)$, $M \mapsto M^\ell = M\omega_X^{-1} = M \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1}$ (here $\omega_X = \Omega_X^{\dim X}$). Therefore $\mathcal{M}(X)$ has the tensor product $M \otimes^! L := (M^\ell \underset{\mathcal{O}_X}{\otimes} L^\ell)\omega_X$. Now the ! tensor product of two $*$ operations is defined according to the following pattern. Let I, J be finite sets, $i_0 \in I$, $j_0 \in J$; denote by $I \vee J$ the disjoint union of I, J with i_0, j_0 identified. Then there is a

[1] We switched to right $\mathcal{D}$-modules to make the "sign rule" for the $*$ operations obvious.

natural map

$$\otimes^{I,J}_{i_0,j_0}: P^*_I(\{M_i\}, L) \otimes P^*_J(\{N_j\}, K) \to P^*_{I\vee J}(\{M_i, N_j, M_{i_0} \otimes^! N_{j_0}\}_{\substack{i\neq i_0\\ j\neq j_0}}, L \otimes^! K)$$

If $I = \{i_0\}$, $J = \{j_0\}$ this is the usual tensor product of morphisms.

A *compound tensor structure* on a category is the data of a tensor and a pseudo-tensor structure related as above by the $\otimes^{I,J}_{i_0,j_0}$ maps which are associative and commutative in the obvious sense. Thus $\mathcal{M}(X)$ (and $\mathcal{M}^\ell(X)$) is a compound tensor category.

0.5. The compound tensor structure allows to do basic differential geometry in $\mathcal{M}(X)$ implementing the principle that the "functions" multiply according to $\otimes^!$ format, while duals and the action of infinitesimal symmetries must be understood in the $*$ sense. In other words, the geometric objects we consider are $\mathcal{D}_X$-schemes, the Lie algebras are actually Lie* algebras, and one defines easily what the action of a Lie* algebra on a $\mathcal{D}_X$-scheme means. Note that an action of a Lie* algebra L on a $\mathcal{D}_X$-scheme $\mathcal{Y}$ yields an action on $\mathcal{Y}$ of the usual Lie algebra $h(L)$.

For example, let G be a group $\mathcal{D}_X$-scheme such that $CoLie(G)$ is a coherent $\mathcal{D}_X$-module. Set $Lie(G) := \mathrm{Hom}_{\mathcal{D}_X}(CoLie(G), \mathcal{D}_X)$ (this usual $\mathcal{D}$-module duality is actually the duality for the $*$ pseudo-tensor structure). Then $Lie(G)$ is a Lie* algebra. A G-action on a $\mathcal{D}_X$-scheme $\mathcal{Y}$ yields a $CoLie(G)$-coaction on $\mathcal{O}_\mathcal{Y}$ (in the $\underset{\mathcal{O}_X}{\otimes}$ sense). Dualizing, we get a $Lie(G)$-action on $\mathcal{Y}$.

0.6. Now a *coisson algebra* is simply a Poisson algebra in the compound setting. Namely, this is a commutative $\mathcal{D}_X$-algebra A^ℓ together with a Lie* algebra structure on A (*the coisson bracket*) such that the adjoint action is compatible with the multiplicative structure on A^ℓ. ¿From this we get an action of $h(A)$ on A^ℓ in the usual sense. Derivations of A^ℓ that come from $h(A)$ are called hamiltonian vector fields. Coisson brackets localize nicely so one knows what is a coisson structure on any $\mathcal{D}_X$-scheme.

0.7. Let $\mathcal{Y} = \mathrm{Spec}\, A^\ell$ be an affine $\mathcal{D}_X$-scheme. The space of horizontal sections $X \to \mathcal{Y}$ is an ind-affine ind-scheme; if X is compact this is actually a scheme. We denote the corresponding (pro) algebra as $\langle A\rangle = H^{ch}_0(X, A)$. The derived version of this construction yields a graded commutative superalgebra $H^{ch}_\cdot(X, A)$.

The above construction also makes sense locally. For example, assume that X is an affine curve; let $X \subset \bar{X}$ be a smooth compactification. For $z \in X^\infty = \bar{X} \setminus X$ the space of horizontal sections of $\mathcal{Y}$ over the (formal) punctured disc at z is again an ind-affine scheme $\operatorname{Spec} A_{(\!(z)\!)}$. The restriction of a global section to the punctured discs defines a surjective morphism $A_{(\!(\infty)\!)} := \widehat{\bigotimes_{X^\infty}} A_{(\!(z)\!)} \twoheadrightarrow H_0^{ch}(X, A)$. Its kernel is the ideal generated by the image of certain canonical map $r_{(\!(\infty)\!)}\colon H^0(X, h(A)) \to A_{(\!(\infty)\!)}$.

If A is a coisson algebra finitely generated as $\mathcal{D}_X$-algebra then $A_{(\!(z)\!)}$ are Poisson algebras, $H^0(X, h(A))$ is a Lie algebra, and $r_{(\!(\infty)\!)}$ commutes with brackets. Therefore $r_{(\!(\infty)\!)}$ is a Hamiltonian action of $H^0(X, h(A))$ on $\operatorname{Spec} A_{(\!(\infty)\!)}$, and $\operatorname{Spec} \langle A \rangle$ is the zero fiber of the momentum map.

0.8. Now let us pass to chiral algebras. There are two complementary (equivalent) approaches to this notion: Lie algebra style and commutative algebra style. We begin with the "Lie algebra" approach; for the "commutative algebra" one see 0.12.

¿From now on we assume that $\dim X = 1$. Let $j^{(I)}\colon U^{(I)} \hookrightarrow X^I$ be the complement to the diagonal divisor. Set

$$P_I^{ch}(\{L_i\}, M) := \operatorname{Hom}(j_*^{(I)} j^{(I)*} \boxtimes L_i, \Delta_*^{(I)} M). \tag{0.8.1}$$

These are *chiral I-operations*. One defines their composition in the obvious manner. We get the *chiral pseudo-tensor structure* on $\mathcal{M}(X)$; the corresponding pseudo-tensor category is denoted by $\mathcal{M}(X)^{ch}$.

Now a *chiral algebra* on X is simply a Lie algebra in $\mathcal{M}(X)^{ch}$ with an additional property (existence of unit), which we prefer to skip at the moment.

0.9. We want to show that the notion of chiral algebra is a quantization of that of coisson algebra. Let us consider first the chiral structure itself.

For a finite non-empty set I denote by $Q(I)$ the set of all quotient sets of I (= equivalence relations on I), so for $S \in Q(I)$ there is a canonical decomposition $I = \coprod_{s \in S} I_s$. The $\mathcal{D}$-module $j_*^{(I)} j^{(I)*} \boxtimes L_i$ on X^I carries a canonical filtration with associated graded module identified with a quotient of $\bigoplus_{S \in Q(I)} \boxtimes_S (\Delta_*^{(I_s)} (\overset{!}{\bigotimes_{I_s}} L_i) \otimes \mathcal{L}ie_{I_s}^*)$ (the Lie operad spaces appear

here as top homology of spaces of configurations of points on a plane, see [C]). Set

$$P^c_I(\{L_i\},M) := \underset{S\in Q(I)}{\oplus} P^*_S(\{\overset{!}{\underset{i\in I_s}{\otimes}} L_i\},M)\otimes(\underset{s\in S}{\otimes}\mathcal{L}ie_{I_s}). \tag{0.9.1}$$

We see that the above filtration on $j^{(I)}_* j^{(I)*}\boxtimes L_i$ yields a filtration on $P^{ch}_I(\{L_i\},M)$ with $\mathrm{gr}P^{ch}_I(\{L_i\},M)$ identified with a subspace of $P^c_I(\{L_i\},M)$; if L_i are locally free $\mathcal{D}_X$-modules and X is affine, then $\mathrm{gr}\,P^{ch} = P^c$. The filtration on P^{ch} is compatible with the composition of chiral operations, so it defines composition of operations from $\mathrm{gr}\,P^{ch}$ and P^c.

Therefore c operations are symbols, or classical limits, of chiral operations. Now it is easy to see that c operations and their composition are defined purely in terms of the compound tensor structure (and, in fact, determine this structure uniquely). So the chiral pseudo-tensor structure is a *quantization* of the compound tensor structure on $\mathcal{M}(X)$.

0.10. Notice that a binary c operation is the same as a pair that consists of a binary ! operation and a binary $*$ operation. One checks easily that such c operation is a Lie bracket if and only if the corresponding ! operation is commutative and associative, and the $*$ operation is a coisson bracket with respect to this commutative algebra structure. Thus Lie^c algebras are the same as coisson algebras.

Now let A be a chiral algebra. The chiral Lie bracket $\mu\in P^{ch}_2(\{A,A\},A)$ ("chiral product") yields a $*$ bracket $[\quad]_\mu\in P^*_2(\{A,A\},A)$, so a chiral algebra is automatically a Lie^* algebra. We say that μ is *commutative* if $[\quad]_\mu=0$. As above, such μ is equivalent to a commutative $\mathcal{D}_X$-algebra structure on A^ℓ. Now assume we have a family $A_\hbar$ of chiral algebras that depend on a parameter $\hbar\in\mathbb{C}$ such that A_0 is commutative. Then $\{\quad\}:=\left(\hbar^{-1}[\quad]_{\mu_\hbar}\right)\big|_{\hbar=0}$ is a coisson bracket on A^ℓ_0. We have shown that chiral algebras are quantizations of coisson algebras as promised.

The problem of quantization of coisson algebras is fairly interesting. We do not know how to solve it in general.[2] In the main body of this work we treat the simplest cases of

[2] Could it be that Kontsevich's formulas can be naturally explained in the framework of chiral algebras?

linear brackets. In particular, we construct chiral enveloping algebras of Lie* algebras and chiral algebras of twisted differential operators.

In a sense, in the "chiral world" chiral algebras are parallel to usual associative algebras while Lie* algebras and commutative $\mathcal{D}_X$-algebras play, respectively, the roles of usual Lie and commutative algebras.

0.11. The constructions from 0.7 generalize to arbitrary chiral algebras. For a chiral algebra A one defines its *chiral homology* $H_i^{ch}(X,A)$. Here $H_0^{ch}(X,A)$ is what mathematical physicists usually call "the space of conformal blocks". For any $z \in X^\infty$ we have an associative algebra $A_{((z))}$; if X is affine then $H_0^{ch}(X,A)$ identifies canonically with a quotient of $A_{((\infty))} = \widehat{\bigotimes_{X^\infty}} A_{((z))}$ modulo a right ideal. In the classical limit we return to the objects from 0.7.

0.12. Let us sketch now the "commutative algebra" style description of chiral algebras. This approach is essential for certain subjects, in particular, for the definition of chiral homology. Let A^ℓ be a left $\mathcal{D}_X$-module. A *factorization algebra* structure on A^ℓ is[3] a datum of left $\mathcal{D}$-modules $A^\ell_{X^n}$ on X^n, $n > 0$, such that for any $(x_1,..,x_n) \in X^n$ the fiber $A^\ell_{(x_1,..,x_n)}$ equals to $\bigotimes_{s\in S} A^\ell_s$ where $S := \{x_1,..,x_n\} \subset X$ (factorization property). We demand that the identifications are compatible with natural symmetries. Therefore $A^\ell_X = A^\ell$, $A^\ell_{X\times X}$ coincides with $A^\ell \boxtimes A^\ell$ over the complement U to the diagonal, and its pull-back to the diagonal equals A^ℓ, etc.

It appears that factorization algebra structure on $\mathcal{A}^\ell$ amounts to chiral algebra structure on $\mathcal{A}$. Namely, the chiral product corresponding to the above factorization algebra structure is the composition $j_*j^*A \boxtimes A = j_*j^*A_{X\times X} \to \Delta_*\Delta^! A_{X\times X} = \Delta_* A$ where the equalities are structure identifications and the arrow is a canonical morphism. Notice that A is commutative if and only if $A^\ell_{X\times X} \supset A^\ell \boxtimes A^\ell$.

0.13. Physicists usually describe the chiral algebra structure in terms of *operator product expansions*, OPE for short (see e.g. [BPZ]). This may be viewed as a generalization of the above description of commutative chiral algebras.

[3] As before, we do not worry about the unit.

To explain what OPE is one needs to consider some non-quasicoherent $\mathcal{D}$-modules. Thus, for a left $\mathcal{D}_X$-module A^ℓ let $\hat{\Delta}_* A^\ell$ be a sheaf of left $\mathcal{D}_{X\times X}$-modules supported on the diagonal $X \hookrightarrow X \times X$ which is I-adically complete and satisfies $\hat{\Delta}_* A^\ell / I\hat{\Delta}_* A^\ell = A^\ell$ (here $I \subset \mathcal{O}_{X\times X}$ is the ideal of the diagonal). Such $\hat{\Delta}_* A^\ell$ exists and is unique. Note that for any local section $a \in A^\ell$ there is a unique section $a^{(1)} \in \hat{\Delta}_* A^\ell$ such that $a^{(1)}$ is horizontal along the second variable and $a^{(1)} \bmod I\hat{\Delta}_* A^\ell = a$. Therefore, if t is a local coordinate on X, one may write any section φ of $\hat{\Delta}_* A^\ell$ as a formal power series $\sum\limits_{i\geq 0} a_i^{(1)}(t_2 - t_1)^i$, $a_i = (i!^{-1}\partial_{t_2}^i \varphi) \bmod I\hat{\Delta}_* A^\ell$. Now let $\tilde{\Delta}_* A^\ell = j_*\mathcal{O}_U \underset{\mathcal{O}_{X\times X}}{\otimes} \hat{\Delta}_* A^\ell$ be localization of $\hat{\Delta}_* A^\ell$ with respect to the equation of the diagonal. A section of $\tilde{\Delta}_* A^\ell$ is a Laurent power series $\sum\limits_{i\gg -\infty} a_i^{(1)}(t_2 - t_1)^i$, i.e. we may write $\tilde{\Delta}_* A^\ell$ as $A^\ell((t_2 - t_1))$.

There is a canonical isomorphism of right $\mathcal{D}_{X\times X}$-modules

$$\mathrm{Hom}_{\mathcal{O}_{X\times X}}(j_*\mathcal{O}_U, \Delta_* A) \xrightarrow{\sim} (\tilde{\Delta}_* A^\ell)\omega_{X\times X}$$

identifying $\varphi\colon j_*\mathcal{O}_U \to \Delta_* A$ with $\sum a_i^{(1)}(t_2 - t_1)^i dt_1 \wedge dt_2$, $a_i dt = \mathrm{tr}_{p_1}(\varphi((t_2 - t_1)^{-i-1}) \in A$; here $\mathrm{tr}_{p_1}\colon p_1.\Delta_* A \to A$ is the canonical trace map for the projection $p_1\colon X \times X \to X$. Therefore, by adjunction, a morphism $\circ\colon A^\ell \boxtimes A^\ell \to \tilde{\Delta}_* A^\ell$ of left $\mathcal{D}_{X\times X}$-modules is the same as a morphism $\mu = \mu_\circ\colon j_* j^* A \boxtimes A = (A \boxtimes A) \underset{\mathcal{O}_{X\times X}}{\otimes} j_*\mathcal{O}_U \to \Delta_* A$ of right $\mathcal{D}$-modules on $X \times X$, i.e., a chiral binary operation (we use the identification $\omega_X \boxtimes \omega_X \xrightarrow[\sim]{} \omega_{X\times X}$, $\nu_1 \boxtimes \nu_2 \mapsto p_1^*\nu_1 \wedge p_2^*\nu_2$). The morphism $\circ = \circ_\mu$, $a \boxtimes b \mapsto \sum (a \underset{i}{\circ} b)^{(1)}(t_2 - t_1)^i$, is the *operator product expansion* disguise of μ.

It is clear what commutativity of $\circ$ means, and it amounts exactly to skew commutativity of $\mu_\circ$. To see what associativity of $\circ$ means take $a, b, c \in A^\ell$ and consider the iterated product

$$a \circ (b \circ c) = \sum_{i\gg -\infty} \sum_{j\gg -\infty} (a \underset{j}{\circ}(b \underset{i}{\circ} c))^{(1)}(t_2 - t_1)^j (t_3 - t_2)^i \in A^\ell((t_2 - t_1))((t_3 - t_2))$$

Informally speaking this series converges in the domain $0 < |t_3 - t_2| \ll |t_2 - t_1| \ll 1$. Just as with μ and $\circ$, one may see that this iterated product amounts exactly to the operation $\mu(id_A, \mu) \in P_3^{ch}(\{A, A, A\}, A)$. Similarly, we may consider $(a \circ b) \circ c \in A^\ell((t_3 - t_1))((t_2 - t_1))$.

Note that $(a \circ b) \circ c$ and $a \circ (b \circ c)$ converge in different domains so to compare them one needs analytic continuation. Precisely, both iterated Laurent power series spaces contain the subspace $A^\ell[\![t_2 - t_1, t_3 - t_2]\!][(t_1 - t_2)^{-1}, (t_1 - t_3)^{-1}, (t_2 - t_3)^{-1}]$, and associativity of $\circ$ means that both iterated products lie in this subspace and coincide (as its elements).

A simple Cousin type argument shows that for commutative $\circ$ its associativity amounts to Jacobi identity for μ, i.e., chiral algebras are the same as commutative and associative OPE algebras.

¿From the point of view of 0.12 OPE is non other than the glueing isomorphism that reconstructs $A^\ell_{X\times X}$ from its restriction to U (equal $A^\ell \boxtimes A^\ell|_U$) and to the formal neighbourhood of the diagonal (equal to $\overset{\wedge}{\Delta}_* A^\ell$).

0.14. A tiny heuristic comment about the origin of chiral algebras in mathematical physics (see [BPZ]). Consider a quantum field theory on X. It unfolds within the algebra of local fields. If we play with an operator version of the theory then the local fields form a sheaf R of functions on X with values in (unbounded) operators acting on a space of states. This R is closed under the action of differential operators with C^∞ coefficients. If $\{x_i\} \in X$ is a finite set of pairwise distinct points on X, then for any local fields r_i the operators $r_i(x_i)$ commute and their product is well-defined.

Denote by $A^\ell \subset R$ the subsheaf of holomorphic local fields (passing to A^ℓ we begin to deal with "holomorphic quantum mechanics" rather than with $2D$ field theory), so A^ℓ is a left $\mathcal{D}$-module on X. Assume further the property that if $x, y \in X$, $x \neq y$, are close points then for $f, g \in A^\ell$ the product $f(x)g(y)$ may be written as Laurent power series $\sum_{i \ll -\infty} a_i(x)(y-x)^i$, $a_i \in A^\ell$. This series is a commutative and associative OPE product on A^ℓ.

0.15. To see an example of a non-commutative chiral algebra let us describe in geometric terms the chiral enveloping algebra of a Kac-Moody Lie algebra.

Let G be an algebraic group. For $x \in X$ consider the set of pairs $(\mathcal{F}, \alpha)$, $\mathcal{F}$ is a G-bundle on X, α is a trivialization of $\mathcal{F}$ on $X \smallsetminus \{x\}$. This is the set of points of a formally smooth ind-scheme $\mathcal{GR}_x$; a choice of a parameter t_x at x identifies $\mathcal{GR}_x$ with $G(\mathbb{C}(\!(t)\!))/G(\mathbb{C}[\![t]\!])$.

Our $\mathcal{GR}_x$ are fibers of an ind-scheme $\mathcal{GR}_X$ over X called the *affine Grassmannian*. This is a $\mathcal{D}_X$-ind-scheme: when x varies infinitesimally $X \setminus \{x\}$, hence $\mathcal{GR}_x$, do not change.

The trivialized G-bundle $(\mathcal{F}, \alpha)$ on X defines the section $e: X \to \mathcal{GR}$ (which is the only horizontal section of $\mathcal{GR}$). Denote by A^ℓ_x the vector space of distributions on $\mathcal{GR}_x$ supported at $e(x)$, i.e., A^ℓ_x is the (topological) dual to the formal completion of the local ring $\mathcal{O}_{\mathcal{GR}_x, e(x)}$. When x varies A^ℓ_x become the fibers of a left $\mathcal{D}_X$-module A^ℓ_X.

Let us show that A is a chiral algebra in a natural way. We follow format 0.12. For any $n > 0$ we have a $\mathcal{D}$-ind-scheme $\mathcal{GR}_{X^n}$ over X^n with fibers $\mathcal{GR}_{(x_1,..,x_n)}$ equal to the space of pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a G-bundle on X, α is a trivialization of $\mathcal{F}$ on $X \setminus \{x_1, ..x_n\}$. As in case $n = 1$ one has a canonical horizontal section e of $\mathcal{GR}_{X^n}$, and we set $A^\ell_{(x_1,..,x_n)}$ to be the vector space of distributions on $\mathcal{GR}_{(x_1,..,x_n)}$ supported at e. These are fibers of a left $\mathcal{D}_{X^n}$-module $A^\ell_{X^n}$. The factorization property for $A^\ell_{X^n}$ follows from the similar property of $\mathcal{GR}_{X^n}$ itself: there is a canonical identification $\mathcal{GR}_{(x_1,..x_n)} = \prod_{s \in S} \mathcal{GR}_s$ where $S = \{x_1, .., x_n\} \subset X$.

To get the chiral envelope of a Kac-Moody algebra of non-zero level one should twist A^ℓ by some canonical line bundle on $\mathcal{GR}$.

0.16. A vertex operator algebra from references in 0.0 is essentially a chiral algebra on the "coordinate disc" $\operatorname{Spec} \mathbb{C}[\![t]\!]$ equipped with an action of the group ind-scheme $\operatorname{Aut} \mathbb{C}[\![t]\!]$ of its symmetries. As usual, such object is the same as a "universal" chiral algebra (= a law that assigns to any curve a chiral algebra on it in a way compatible with étale localization). Similarly, "universal" Lie* algebras are the same as objects dubbed "conformal algebras" in the book [K].

In some versions of VOA axioms it is assumed that the action of the Lie algebra $\operatorname{Der} \mathbb{C}[\![t]\!]$ comes from a stress-energy tensor which amounts to a morphism from Virasoro Lie* algebra to our chiral algebra.

0.17. Our work contains no new deep results. Its mere purpose was to understand the ingenious formal power series manipulations that haunt the books on vertex operator algebras.

A really challenging problem is to find out what is a chiral algebra on higher dimensional X (recall that coisson algebras live in any dimension). The definitions from 0.9, 0.10 formally work also for $\dim X > 1$, but to make them sensible one must plunge into derived categories. This brings the uneasiness which we had no courage to overcome.

0.18. The work consists of three chapters. In the first one we discuss some relevant abstract nonsense. The classical (coisson) part of the picture is treated in the second chapter. The third chapter deals with chiral algebras proper.

As usual, distilled axioms are pretty indigestible. We do suggest the reader to follow complicated alembics of Chapter One simultaneously with respective sections of more wholesome Chapters Two and Three.

0.19. Our sincere gratitude is due to Sasha Belavin and Pierre Deligne. Apart from an obvious influence of their ideas, it was in their apartments where the prime part of this work was done. The first author would also like ot thank IHES, MPI, and IAS for their hospitality, and his wife for her being able to endure the rewritings of the draft. We are grateful to Dottie Phares and Richard A. Lloyd for the careful typing of the first part of the first version of the manuscript.

0.20. A few words about notation commonly used below. "$x \in Y$" means that either x is an element of a set Y, or x is an object of a category Y, or x is a local section of a sheaf Y. For a category $\mathcal{M}$ we denote by $\mathcal{M}^\circ$ the dual category; $Sets$ denotes the category of all sets. For a smooth variety X we denote by Θ_X and $\mathcal{D}_X$ the tangent sheaf and the algebra of differential operators.

CHAPTER ONE: AXIOMATIC PATTERNS

Да поразит тя пуще грома
Ужасна, сильна аксиома.

А.А.Нахимов, "Поэт и Математик",
Харьковский Демокрит, 1816 +

§1 Pseudo-tensor categories

1.1.1. Denote by $\mathcal{S}$ the category of finite non-empty sets and surjective maps. For a morphism $\pi: J \twoheadrightarrow I$ in $\mathcal{S}$ and $i \in I$ set $J_i := \pi^{-1}(i) \subset J$; let $\cdot \in \mathcal{S}$ be the one element set.

Definition. [4] A *pseudo-tensor category* is a class of objects $\mathcal{M}$ together with the following data:

(a) For any $I \in \mathcal{S}$, an I-family of objects $L_i \in \mathcal{M}$, $i \in I$, and an object $M \in \mathcal{M}$ one has the set $P_I^{\mathcal{M}}(\{L_i\}, M) = P_I(\{L_i\}, M)$; its elements are called I-operations.

(b) For any map $\pi: J \twoheadrightarrow I$ in $\mathcal{S}$, families of objects $\{L_i\}_{i\in I}$, $\{K_j\}_{j\in J}$ and an object M one has the composition map

$$P_I(\{L_i\}, M) \times \prod_I P_{J_i}(\{K_j\}, L_i) \to P_J(\{K_j\}, M)$$

$(\varphi, (\psi_i)) \mapsto \varphi(\psi_i)$. Here in the notation $P_{J_i}(\{K_j\}, L_i)$ we assume that $j \in J_i$.

+ Let thee be smitten as by thunderbolt
By a dreadful, strong axiom.
A.A. Nakhimov, "A Poet and a Mathematician",
Kharkovski Democritus, 1816

[4] This kind of objects were considered sporadically in mathematical literature under different names. See e.g. [La] ("multicategories") or [Li] ("multilinear categories").

These data should satisfy the following properties.

(i) The composition is associative: if $H \twoheadrightarrow J$ is another surjective map,$\{F_h\}$ is an H-family of objects, $\chi_j \in P_{H_j}(\{F_h\}, K_j)$ then $\varphi(\psi_i(\chi_j)) = (\varphi(\psi_i))(\chi_j) \in P_H(\{F_h\}, M)$.

(ii) For any $M \in \mathcal{M}$ there is an element $id_M \in P_\cdot(\{M\}, M)$ such that for any $\varphi \in P_I(\{L_i\}, M)$ one has $id_M(\varphi) = \varphi(id_{L_i}) = \varphi$. $\square$

We will often use the notation $P_{J/I}(\{K_j\}, \{L_i\}) = P_\pi(\{K_j\}, \{L_i\})$ for $\prod\limits_I P_{J_i}(\{K_j\}, L_i)$ to write the composition map as $P_I \times P_{J/I} \to P_J$.

1.1.2. Let $\mathcal{M}$ be a pseudo-tensor category. If $I = \{1, \dots, n\}$ we write P_n for P_I. For $L, M \in \mathcal{M}$ set $\mathrm{Hom}(L, M) = P_1(\{L\}, M)$. The composition of morphisms comes from (b), therefore $\mathcal{M}$ becomes a usual category. The composition with morphisms makes P_I a functor $(\mathcal{M}^\circ)^I \times \mathcal{M} \to Sets$. Therefore, we may consider the pseudo-tensor categories as usual categories equipped with extra *pseudo-tensor structure* formed by a collection of functors P_I, $|I| \geq 2$, together with composition morphisms.

1.1.3. Assume that a family $\{L_i\}_{i \in I}$ has the property that the functor $M \mapsto P_I(\{L_i\}, M)$ is representable. We denote the corresponding object of $\mathcal{M}$ as $\underset{I}{\otimes} L_i$ and call it the *pseudo-tensor product* of L_i's. If the pseudo-tensor product exists for any family of objects, then we call our pseudo-tensor structure *representable*. In other words, a representable pseudo-tensor structure on a category $\mathcal{M}$ is a rule that assigns to any $I \in \mathcal{S}$ a functor $\underset{I}{\otimes}: \mathcal{M}^I \to \mathcal{M}$ and to any map $\pi: J \twoheadrightarrow I$ a natural compatibility morphism

$$\varepsilon_\pi: \underset{J}{\otimes} K_j \to \underset{I}{\otimes}(\underset{J_i}{\otimes} K_j)$$

The morphisms ε_π must behave naturally with respect to composition of the maps π, and $\otimes$ is the identity functor.

If all the compatibility morphisms ε_π are isomorphisms, then we arrive to the usual notion of tensor (= symmetric monoidal) category.

1.1.4. A pseudo-tensor category $\mathcal{B}$ with single object is the same as an *operad* (see e.g. [May], [L], [GK], [Kap]). Namely, such $\mathcal{B}$ amounts to a collection of sets $\mathcal{B}_n = P_n^{\mathcal{B}}$, $n \geq 1$,

equipped with the action of symmetric group Σ_n, together with appropriate "composition of operations" maps between them, and our axioms coincide here with those of an operad.

Example. The linear orders operad Σ: one has $\Sigma_I = \{\text{linear orders on } I\}$, the composition is defined by the lexicographic rule.

1.1.5. Let $\mathcal{M}$, $\mathcal{N}$ be pseudo-tensor categories. A *pseudo-tensor functor* $\tau: \mathcal{N} \to \mathcal{M}$ is a law that assigns to any object $G \in \mathcal{N}$ an object $\tau(G) \in \mathcal{M}$ and to $\{F_i\}_{i\in I}$, $G \in \mathcal{N}$ a map $\tau_I: P_I^{\mathcal{N}}(\{F_i\}, G) \to P_I^{\mathcal{M}}(\{\tau(F_i)\}, \tau(G))$ so that τ_I are compatible with composition and $\tau.(id_G) = id_{\tau(G)}$. One defines a morphism between pseudo-tensor functors in the obvious way, therefore the pseudo-tensor functors $\mathcal{N} \to \mathcal{M}$ form a category. The composition of pseudo-tensor functors is a pseudo-tensor functor in the obvious way.

Any pseudo-tensor functor τ defines a usual functor between the corresponding usual categories. We call τ a *pseudo-tensor extension* of this usual functor.

An *adjoint pair* is a pair of pseudo-tensor functors $\mathcal{M} \underset{\tau}{\overset{\nu}{\rightleftarrows}} \mathcal{N}$ together with identifications $a: P_I^{\mathcal{N}}(\{\nu(M_i)\}, G) \xrightarrow{\sim} P_I^{\mathcal{M}}(\{M_i\}, \tau(G))$ for $M_i \in \mathcal{M}$, $G \in \mathcal{N}$, compatible with composition of operations.[5] We say that ν is left adjoint to τ, and τ is right adjoint to ν. The corresponding usual functors are obviously adjoint.

Exercise. Show that for a given pseudo-tensor functor its left (resp. right) adjoint pseudo-tensor functor is determined uniquely up to a unique isomorphism (if it exists).

1.1.6 Examples. (i) A *pseudo-tensor subcategory* of a pseudo-tensor category $\mathcal{M}$ is given by a subclass of objects $\mathcal{T} \subset \mathcal{M}$ together with subsets $P_I^{\mathcal{T}}(\{L_i\}, M) \subset P_I^{\mathcal{M}}(\{L_i\}, M)$ for L_i, $M \in \mathcal{T}$ such that $P_I^{\mathcal{T}}$ are closed under the composition of operations, and $id_M \in P_{\cdot}^{\mathcal{T}}(\{M\}, M)$ for any $M \in \mathcal{T}$. Then $(\mathcal{T}, P^{\mathcal{T}})$ is a pseudo-tensor category, and $\mathcal{T} \hookrightarrow \mathcal{M}$ is a pseudo-tensor functor. If $P_I^{\mathcal{T}}(\{L_i\}, M) = P_I^{\mathcal{M}}(\{L_i\}, M)$ for any L_i, $M \in \mathcal{P}$, then we call $\mathcal{T}$ a *full* pseudo-tensor subcategory of $\mathcal{M}$.

Any subclass of objects of $\mathcal{M}$ may be considered as a full pseudo-tensor subcategory of $\mathcal{M}$. In particular, for any $M \in \mathcal{M}$ we have the full pseudo-tensor subcategory with single

[5] Precisely, we demand that for $\varphi \in P_I^{\mathcal{N}}(\{\nu(M_i)\}, G)$, $\psi_i \in P_{J_i}^{\mathcal{M}}(\{L_j\}, M_i)$ one has $a(\varphi(\nu(\psi_i))) = a(\varphi)(\psi_i)$ and for $\chi \in P_K^{\mathcal{N}}(\{G_k\}, H)$, $\varphi_k \in P_{I_k}^{\mathcal{N}}(\{\nu(M_i)\}, G_k)$ one has $a(\chi(\varphi_k)) = \tau(\chi)(a(\varphi_k))$.

object M. We denote by $P(M) = P^{\mathcal{M}}(M)$ the corresonding operad.

(ii) Assume that $\mathcal{M}$, $\mathcal{N}$ have representable pseudo-tensor structures. Then a pseudo-tensor functor $\mathcal{N} \to \mathcal{M}$ is the same as a usual functor $\tau: \mathcal{N} \to \mathcal{M}$ together with natural morphisms $\nu_I: \underset{I}{\otimes} \tau(N_i) \to \tau(\underset{I}{\otimes} N_i)$ for any finite families of objects $N_i \in \mathcal{N}$, such that ν_I commute with the compatibility morphisms ε_π (i.e., for any $\pi: J \twoheadrightarrow I$ and a J-family $K_j \in \mathcal{N}$ the diagram

$$\begin{array}{ccc}
\underset{J}{\otimes}\tau(K_j) & \xrightarrow{\nu_J} & \tau(\underset{J}{\otimes}K_j) \\
\downarrow{\scriptstyle \varepsilon_\pi^{\mathcal{M}}} & & \\
\underset{I}{\otimes}(\underset{J_i}{\otimes}\tau(K_j)) & & \downarrow{\scriptstyle \tau(\varepsilon_\pi^{\mathcal{N}})} \\
\downarrow{\scriptstyle \underset{I}{\otimes}\nu_{J_i}} & & \\
\underset{I}{\otimes}\tau(\underset{J_i}{\otimes}K_j) & \xrightarrow{\nu_I} & \tau(\underset{I}{\otimes}(\underset{J_i}{\otimes}K_j))
\end{array}$$

commutes). In particular, any tensor functor between tensor categories is a pseudo-tensor functor.

(iii) If $\mathcal{B}$ is an operad then we call a pseudo-tensor functor $\mathcal{B} \to \mathcal{M}$ a $\mathcal{B}$ *algebra* in $\mathcal{M}$; we denote the category of $\mathcal{B}$ algebras in $\mathcal{M}$ as $\mathcal{B}(\mathcal{M})$. A $\mathcal{B}$ algebra in $\mathcal{M}$ amounts to an object $L \in \mathcal{M}$ together with a morphism of operads $\mathcal{B} \to P^{\mathcal{M}}(L)$.

For example, a Σ algebra in $\mathcal{M}$ is just a monoid, i.e., an object L equipped with an associative product $\cdot \in P_2(\{L, L\}, L) = P^{\mathcal{M}}(L)_2$. An algebra for a "trivial" one point operad is a commutative monoid.

(iv) For any family $\{\mathcal{M}_j\}$ of pseudo-tensor categories one defines their product $\Pi\mathcal{M}_j$ in the obvious way. A pseudo-tensor functor $\mathcal{N} \to \Pi\mathcal{M}_j$ is the same as a collection of pseudo-tensor functors $\mathcal{N} \to \mathcal{M}_j$.

(v) If $\mathcal{A}$ is a tensor category then the product functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a pseudo-tensor functor in the obvious way. Let $\mathcal{M}$ be a pseudo-tensor category. An *$\mathcal{A}$-Action on $\mathcal{M}$* is a pseudo-tensor functor $\otimes: \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ together with an identification of two pseudo-tensor functors $\mathcal{A} \times \mathcal{A} \times \mathcal{M} \to \mathcal{M}$, $(A_1 \otimes A_2) \otimes M \xrightarrow{\sim} A_1 \otimes (A_2 \otimes M)$, that satisfies the obvious compatibilities. In concrete terms, such Action is given by an Action of $\mathcal{A}$ on $\mathcal{M}$ as on a

usual category together with natural maps $P_I^{\mathcal{M}}(\{M_i\}, N) \to P_I^{\mathcal{M}}(\{A_i \otimes M_i\}, (\otimes A_i) \otimes N)$ for $M_i, N \in \mathcal{M}$, $A_i \in \mathcal{A}$.

1.1.7. The above definitions have an obvious generalization when operations $P_I(\{L_i\}, M)$ are not sets but objects of a certain fixed tensor (or even pseudo-tensor!) category $\mathcal{A}$.[6] We call such thing a *pseudo-tensor $\mathcal{A}$-category.* The common example is $\mathcal{A}$ = the tensor category of k-modules, k is a commutative ring; this is the case of *pseudo-tensor k-categories.* Other examples (super setting, DG categories, etc.) are discussed in 1.1.16.

Explicitly, a pseudo-tensor k-category is a pseudo-tensor category $\mathcal{M}$ together with a k-module structure on the sets P_I such that the composition map is k-polylinear. For example, if $\mathcal{M}$ consists of a single object then we have a k-operad. A pseudo-tensor k-category $\mathcal{M}$ is *additive* if it is additive as a usual k-category, and $\mathcal{M}$ is *abelian* if it is abelian as a usual k-category and the polylinear functors P_I are left exact.

A pseudo-tensor functor $F\colon \mathcal{N} \to \mathcal{M}$ between the pseudo-tensor k-categories is *k-linear* if the maps F_I are k-linear. If $\mathcal{B}$ is a k-operad we get a notion of a $\mathcal{B}$ k-algebra in $\mathcal{M}$ (we usually call them simply $\mathcal{B}$ algebras).

For example, we have the favorite k-operads $\mathcal{A}ss$, $\mathcal{C}om$, $\mathcal{L}ie$, $\mathcal{P}ois$ which give rise, respectively, to associative, commutative and associative, Lie, and Poisson algebras. Note that $\mathcal{A}ss$, $\mathcal{C}om$ are k-envelopes of the corresponding set-theoretic operads: $\mathcal{A}ss = k[\Sigma]$, $\mathcal{C}om = k$. To avoid problems with skew-symmetry we always assume that $1/2 \in k$ when speaking about Lie or Poisson algebras.

1.1.8 Remark. Let $\mathcal{B}$ be any k-operad. For $I \in \mathcal{S}$ denote by $\mathcal{FB}(I)$ the free $\mathcal{B}$ algebra in the tensor category k mod of k-modules with generators e_i labeled by elements $i \in I$. Then the morphism of k-modules $\mathcal{B}_I \to \mathcal{FB}(I)$, $\varphi \mapsto \varphi(\bigotimes_I e_i)$, is injective, so one may consider $\mathcal{B}_I$ as the k-submodule of $\mathcal{FB}(I)$ generated by "multiplicity free monomials of degree I".

1.1.9. Let $\mathcal{N}_j$, $j \in J$, be a finite family of pseudo-tensor k-categories. Their *tensor product* $\otimes \mathcal{N}_j$ is a pseudo-tensor category defined as follows. Its objects are families of objects

[6] So the composition maps are morphisms $P_I \otimes (\otimes_I P_{J_i}) \to P_J$ in $\mathcal{A}$, etc.

$N_j \in \mathcal{N}_j$; we write the corresponding object of $\underset{J}{\otimes} \mathcal{N}_j$ as $\otimes N_j$. One has $P_I(\{\otimes N_{j_i}\}, \otimes K_j) := \underset{kJ}{\otimes} P_I^{\mathcal{N}_j}(\{N_{j_i}\}, K_j)$; the composition maps are tensor products of those in $\mathcal{N}_j$. Note that for any pseudo-tensor k-category $\mathcal{M}$ a k-linear pseudo-tensor functor $\otimes \mathcal{N}_j \to \mathcal{M}$ is the same as a k-polylinear pseudo-tensor functor $\Pi \mathcal{N}_j \to \mathcal{M}$.

1.1.10. Let $\mathcal{M}$ be any pseudo-tensor k-category. Consider the pseudo-tensor k-category $\mathcal{M} \otimes \mathcal{L}ie$. It coincides with $\mathcal{M}$ as usual k-category and one has $P_I^{\mathcal{M}\otimes\mathcal{L}ie}(\{L_i\}, M) = P_I^{\mathcal{M}}(\{L_i\}, M) \otimes \mathcal{L}ie_I$. The following funny lemma will be of use.

Lemma. $\mathcal{L}ie(\mathcal{M} \otimes \mathcal{L}ie) = \mathcal{C}om(\mathcal{M})$.

Proof. We have to show that Lie algebras in $\mathcal{M} \otimes \mathcal{L}ie$ are the same as commutative algebras in $\mathcal{M}$. Let M be an object in $\mathcal{M}$. One has $P^{\mathcal{M}\otimes\mathcal{L}ie}(M)_2 = P^{\mathcal{M}}(M)_2 \otimes \mathcal{L}ie_2$, so a skew-symmetric binary $\mathcal{M} \otimes \mathcal{L}ie$ operation $[\ \ ]_M$ on M amounts to a symmetric binary $\mathcal{M}$ operation $\cdot_M$. Let us identify $\mathcal{L}ie_n$ with the space of Lie polynomials of variables $e_1, \dots, e_n$, so $[\ \ ]_M = \cdot_M \otimes [e_1, e_2]$. Since $\mathcal{L}ie_3$ is generated by $[e_1, [e_2, e_3]]$, $[e_3, [e_1, e_2]]$, $[e_2, [e_3, e_1]]$ with the only relation $[e_1, [e_2, e_3]] + [e_3, [e_1, e_2]] + [e_2, [e_3, e_1]] = 0$ we see that $[\ \ ]_M$ satisfies Jacobi relation iff $\cdot_M$ is associative. $\square$

1.1.11. Remark (will not be used in the sequel). V. Ginzburg pointed out that the above lemma is a particular case of a general duality statement for quadratic operads.

To formulate it we need some notions from [GK] §2. Assume for simplicity that k is a field. Let $\mathcal{A}$ be a quadratic k-operad (i.e., $\mathcal{A}$ is a quotient of the free k-operad generated by a finite dimensional vector space with Σ_2-action $\mathcal{A}_2$ modulo relations in $\mathcal{A}_3$). Its quadratic dual $\mathcal{A}^!$ is defined in [GK] 2.1.9 (this is a quadratic operad with $\mathcal{A}^!_2$ equal to the vector space dual to $\mathcal{A}_2$, the Σ_2-action is twisted by sgn). If $\mathcal{B}$ is another quadratic operad then we have a quadratic operad $\mathcal{A} \bullet \mathcal{B}$ defined in [GK] 2.2.5 (one has $(\mathcal{A} \bullet \mathcal{B})_2 = \mathcal{A}_2 \otimes \mathcal{B}_2$ with the Σ_2-action twisted by sgn) .

Notice that for any k-operad $\mathcal{P}$ the map $\mathrm{Hom}(\mathcal{A}, \mathcal{P}) \to \mathrm{Hom}(\mathcal{A}_2, \mathcal{P}_2)$ is injective. Let $\mathcal{P}^{quad}$ be a quadratic operad with $\mathcal{P}_2^{quad} = \mathcal{P}_2$ and relations coming from $\mathcal{P}_3$. Then the obvious morphism $\mathcal{P}^{quad} \to \mathcal{P}$ yields a bijection $\mathrm{Hom}(\mathcal{A}, \mathcal{P}^{quad}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{A}, \mathcal{P})$.

Lemma. For any quadratic operads $\mathcal{A}, \mathcal{B}$ and a pseudo-tensor k-category $\mathcal{M}$ one has

$$\mathcal{A}(\mathcal{M} \otimes \mathcal{B}) = (\mathcal{A} \bullet \mathcal{B}^!)(\mathcal{M}) = \mathcal{B}^!(\mathcal{M} \otimes \mathcal{A}^!). \tag{1.1.11.1}$$

Proof. Since $\bullet$ is symmetric it suffices to define the first identification. We may assume that $\mathcal{M}$ has single object. So we have a k-operad $\mathcal{P}$, and we want to define a canonical bijection $\mathrm{Hom}(\mathcal{A}, \mathcal{P} \otimes \mathcal{B}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{A} \bullet \mathcal{B}^!, \mathcal{P})$. It comes from the obvious identification $\mathrm{Hom}(\mathcal{A}_2, (\mathcal{P} \otimes \mathcal{B})_2) = \mathcal{A}_2^! \otimes \mathcal{P}_2 \otimes \mathcal{B}_2 = \mathrm{Hom}((\mathcal{A} \bullet \mathcal{B}^!)_2, \mathcal{P}_2)$. To check the relations we may replace $\mathcal{P}$ by $\mathcal{P}^{quad}$, $\mathcal{P} \otimes \mathcal{B}$ by $(\mathcal{P} \otimes \mathcal{B})^{quad} = \mathcal{P}^{quad} \circ \mathcal{B}$ (see [GK] 2.2.3), and we are done by [GK] 2.2.6(b). □

If $\mathcal{A} = \mathcal{B} = \mathcal{C}om$ then $\mathcal{A}^! = \mathcal{B}^! == \mathcal{L}ie$ (see [GK] 2.1.11) and the above lemma becomes the lemma from 1.1.10.

1.1.12 Remark. Assume that our $\mathcal{M}$ is actually a tensor category with tensor product $\otimes$. Let us show that the (pseudo) tensor structure of $\mathcal{M}$ reconstructs uniquely from that of $\mathcal{M} \otimes \mathcal{L}ie$.

Note that the pseudo-tensor structure of $\mathcal{M} \otimes \mathcal{L}ie$ is representable; denote by $\otimes^{Lie}$ the corresponding tensor product. One has $\underset{I}{\otimes}^{Lie} = \mathcal{L}ie_I^* \otimes \underset{I}{\otimes}$. So one has a canonical identification $M_1 \otimes M_2 = M_1 \otimes^{Lie} M_2$, and the commutativity constraints differ by sign. It remains to recover the associativity constraint. For $M_1, M_2, M_3 \in \mathcal{M}$ consider the composition $\underset{\{1,2,3\}}{\otimes}^{Lie} M_i \to M_1 \otimes^{Lie} (M_2 \otimes^{Lie} M_3) \xrightarrow{\sim} M_1 \otimes (M_2 \otimes M_3)$; here the first arrow is compatibility morphism for $\otimes^{Lie}$ (see 1.1.3), and the second isomorphism comes from the ordering of our indices (e.g., $Lie_{\{2,3\}} \xrightarrow{\sim} k$, $[e_2, e_3] \mapsto 1$). Let $\underset{\{1,2,3\}}{\otimes}^{Lie} M_i \to M_1 \otimes (M_2 \otimes M_3) \oplus M_3 \otimes (M_1 \otimes M_2) \oplus M_2 \otimes (M_3 \otimes M_1)$ be the sum of these maps for cyclic transposition of indices; denote by K its cokernel. The projection maps each summand to K isomorphically, and the associativity constraint for $\otimes$ coincides with composition $M_1 \otimes (M_2 \otimes M_3) \xrightarrow{\sim} K \xrightarrow{\sim} M_3 \otimes (M_1 \otimes M_2) = (M_1 \otimes M_2) \otimes M_3$.

1.1.13. One may use operads to define pseudo-tensor structures on some categories of modules and their dual categories. Let us explain how to do this in the k-linear setting.

Let R be an associative k-algebra; denote by $R\mathrm{mod}$, $\mathrm{mod}R$, the category of left (right) R-modules. Let $\mathcal{B}$ be a *strict R-operad*, i.e., a k-operad endowed with an isomorphism of

k-algebras $R \xrightarrow{\sim} \mathcal{B}_1$. So for any $I \in \mathcal{S}$ the k-module $\mathcal{B}_I$ is actually an $(R - R^{\otimes I})$-bimodule (here $\otimes$ is tensor power over k), and for $J \twoheadrightarrow I$ the composition map may be written as a morphism of $(R - R^{\otimes J})$-bimodules $\mathcal{B}_I \underset{R^{\otimes I}}{\otimes} \mathcal{B}_{J/I} \to \mathcal{B}_J$.

Such $\mathcal{B}$ defines the pseudo-tensor structures $P^* = P^{*\mathcal{B}}$ and $P^! = P^{!\mathcal{B}}$ on mod$R$ and Rmod$^\circ$, respectively. Namely, for $L_i, M \in \mathrm{mod}R$ set

$$P^*_I(\{L_i\}, M) := \mathrm{Hom}_{\mathrm{mod}\, R^{\otimes I}}\left(\otimes L_i, M \underset{R}{\otimes} \mathcal{B}_I\right)$$

The composition of $*$ operations comes from the composition law in $\mathcal{B}$. Similarly, for C_i, $D \in R$mod set

$$P^!_I(D, \{C_i\}) := \mathrm{Hom}_{R\mathrm{mod}}\left(D, \mathcal{B}_I \underset{R^{\otimes I}}{\otimes} (\underset{I}{\otimes} C_i)\right)$$

here we write the operations in Rmod$^\circ$ as operations on R-modules with reversed order of arguments, so that $P^!_\cdot(D, \{C\}) = \mathrm{Hom}_{R\mathrm{mod}}(D, C)$ $(= \mathrm{Hom}_{R\mathrm{mod}^\circ}(C, D))$. The composition of ! operations comes in an obvious manner from the composition law in $\mathcal{B}$.

Remarks. (i) $\mathcal{B} = P^*(R) = P^!(R)$; here we consider R as right and left R-module, respectively.

(ii) The $*$ and ! structures are abelian (see 1.1.7) iff $\mathcal{B}_I$ are flat as, respectively, R- and $R^{\otimes I}$-modules.

1.1.14 Example. Assume that R has a cocommutative Hopf algebra structure. It defines a strict R-operad $\mathcal{B}^R$. Namely, $\mathcal{B}^R_I := R^{\otimes I}$, and the composition map $\mathcal{B}^R_I \otimes \mathcal{B}^R_{J/I} \to \mathcal{B}_J$ is $(\underset{I}{\otimes} r_i) \otimes \underset{I}{\otimes} (\underset{J_i}{\otimes} r^{(i)}_j) \mapsto \underset{I}{\otimes} (\Delta^{(J_i)}(r_i) \cdot (\underset{J_i}{\otimes} r^{(i)}_j)) \in \underset{I}{\otimes} R^{\otimes J_i} = R^{\otimes J}$. Here $\Delta^{(J_i)}: R \to R^{\otimes J_i}$ is the J_i-multiple coproduct. Note that the ! pseudo-tensor structure coincides with the standard tensor category structure on Rmod.

1.1.15. The above notions easily sheafify. Namely, let X be a topological space (or a Grothendieck topology) and $\mathcal{M}$ a sheaf of categories on X (we assume that both morphisms and objects of $\mathcal{M}$ have local nature, i.e., satisfy the glueing property). A *pseudo-tensor structure* on $\mathcal{M}$ assigns to any open U a pseudo-tensor structure on $\mathcal{M}(U)$ and to any $j: V \to U$ a pseudo-tensor extension of the pull-back functor $j^*: \mathcal{M}(U) \to \mathcal{M}(V)$. We

assume that this extension is compatible with composition of j's and the pseudo-tensor operations have local nature (i.e., for any L_i, $M \in \mathcal{M}(U)$ the prehseaf $\underline{P}_I(\{L_i\}, M)$ on U, $\underline{P}_I(\{L_i\}, M)(V) = P_I(\{L_i\big|_V\}, M\big|_V)$, is actually a sheaf). We call such $\mathcal{M}$ a *sheaf of pseudo-tensor categories*. All the notions we discuss in Chapter 1 have straightforward sheaf versions. We will tacitly assume them for the next chapters.

1.1.16. Sometimes we need to consider superobjects in a given category. The following "super" conventions are in force for this work:

Let $\mathcal{M}$ be a k-category. Its "super"(:= $\mathbb{Z}/2$-graded) objects form a k-category $\mathcal{M}^s$. This is also a "super" category: for $M, N \in \mathcal{M}^s$ the k-module of all $\mathcal{M}$-morphisms $\mathrm{Hom}(M, N)$ has an obvious $\mathbb{Z}/2$-grading, and morphisms in $\mathcal{M}^s$ is the even part in $\mathrm{Hom}(M, N)$. If $\mathcal{M}$ is a tensor category then $\mathcal{M}^s$ is automatically a tensor (super) category according to the Koszul rule of signs.

A $\mathbb{Z}$-graded superobject M in $\mathcal{M}$ is a superobject in $\mathcal{M}$ equipped with an extra $\mathbb{Z}$-grading. For such M we set $M[1] = M \otimes k[1]$ where $k[1]$ is an odd copy of k placed in degree -1, i.e., $M[1]$ is the $\mathbb{Z}$-graded super object obtained from M by simultaneous shift of $\mathbb{Z}$- and $\mathbb{Z}/2$-gradings by 1. The shift is an automorphism of the category of $\mathbb{Z}$-graded superobjects, so we have $M[a]$ for every $a \in \mathbb{Z}$. We consider M^a (*not* $M^a[-a]$!) as a subobject of M. The category $\mathcal{M}^{s\mathbb{Z}}$ of A $\mathbb{Z}$-graded superobjects has an obvious "$\mathbb{Z}$-graded superstructure": for $M, N \in \mathcal{M}^{s\mathbb{Z}}$ we have a $\mathbb{Z}$-graded super k-module $\mathrm{Hom}(M, N)$ with components $\mathrm{Hom}(M, N)^i = \Pi\mathrm{Hom}(M^a, N^{a+i})$. If $\mathcal{M}$ is a tensor category then such is $\mathcal{M}^{s\mathbb{Z}}$ (the signs come from the "super" grading, so the forgetting of the $\mathbb{Z}$-grading is a tensor functor).

A DG superobject in $\mathcal{M}$ is a $\mathbb{Z}$-graded superobject equipped with an action of the commutative graded Lie superalgebra $k[-1]$, which is the same as an odd endomorphism d of degree 1 with square 0. DG super objects form a DG super category $\mathcal{M}^{sDG}$ in the obvious manner. This is a tensor DG super category if $\mathcal{M}$ is a tensor category.

If $\mathcal{M}$ is a pseudo-tensor k-category then $\mathcal{M}^s$ is automatically a pseudo-tensor super k-category (see 1.1.7). Namely, for $L_i, M \in \mathcal{M}^s$ the k-module $P_I(\{L_i\}, M)$ carries the obvious

$(\mathbb{Z}/2)^I \times \mathbb{Z}/2$-grading, hence it is a super k-module with respect to the total $\mathbb{Z}/2$-grading. The composition takes into account the Koszul sign rule. Thus for any $L_i, M \in \mathcal{M}^s$, and free super k-modules V_i, U of finite rank one has a canonical identification of super k-modules $P_I(\{L_i \otimes V_i\}, M \otimes U) = P_I(\{L_i\}, M) \otimes \mathrm{Hom}(\underset{I}{\otimes} V_i, U)$.[7] One can also view $\mathcal{M}^s$ as a plain pseudo-tensor k-category considering only even operations.

Similarly, $\mathcal{M}^{s\mathbb{Z}}$ is pseudo-tensor $\mathbb{Z}$-graded super k-category, and $\mathcal{M}^{sDG}$ is a pseudo-tensor DG super k-category. We have a canonical identification $P_I(\{L_i[a_i]\}, M[b]) = P(\{L_i\}, M)[b - \Sigma a_i]$.

§2 Complements

1.2.1 Inner Hom. Let $\mathcal{M}$ be a pseudo-tensor category, I a finite set, $\{L_i\}_{i\in I}$, M objects in $\mathcal{M}$. Assume we have an object $X \in \mathcal{M}$ and an element $\varepsilon \in P_{\widetilde{I}}(\{L_i, X\}, M)$ where $\widetilde{I} := I \sqcup \cdot$. Then for any $J \in \mathbb{S}$ and a J-family of objects $\{A_j\}$ we get a natural map

$$\kappa = \kappa_J(\varepsilon): P_J(\{A_j\}, X) \to P_{J\sqcup I}(\{A_j, L_i\}, M) \tag{1.2.1.1}$$

$\kappa(\varphi) := \varepsilon(\varphi, id_{L_i})$. Here "natural" means that for any $H \twoheadrightarrow J$, an H-family of objects $\{B_h\}$ and $\chi_j \in P_{H_j}(\{B_h\}, A_j)$ one has $\kappa(\varphi(\chi_j)) = \kappa(\varphi)(\chi_j, id_{L_i})$. A pair (X, ε) with the property that all the maps κ are isomorphisms is uniquely defined (if its exists); we denote it $(\mathcal{P}_I(\{L_i\}, M), \varepsilon_{I,\{L_i\},M})$ and call *inner P object*. Note that $\mathcal{P}_\phi(M) = M$, and $\mathcal{P}_1(L, M) = \mathcal{H}om(L, M)$ is the "inner Hom" object.

If well-defined, inner P objects depend on their arguments in functorial manner: for a surjection $E \twoheadrightarrow I$, $\varphi \in \mathrm{Hom}(M, N)$, and $\psi_i \in P_{E_i}(\{K_e\}, L_i)$ (as usually $E_i := \pi^{-1}(i)$) there is an obvious morphism $\mathcal{P}_I(\{L_i\}, M) \to \mathcal{P}_E(\{K_e\}, N)$ which is compatible with composition of φ's and ψ's.

1.2.2. Here is an "inner" version of the functoriality property. Let $\pi: E \to I$ be *any* map of finite sets, $\{K_e\}$ an E-family of objects. Assume that the objects $\mathcal{P}_I(\{L_i\}, M)$, $\mathcal{P}_E(\{K_e\}, M)$, $\mathcal{P}_{E_i}(\{K_e\}, L_i)$ exist. Then one has a canonical "composition" element

$$c \in P_{\widetilde{I}}(\{\mathcal{P}_I(\{L_i\}, M),\ \mathcal{P}_{E_i}(\{K_e\}, L_i)\},\ \mathcal{P}_E(\{K_e\}, M)) \tag{1.2.2.1}$$

[7] and this identification is compatible with composition of operations.

defined as the image of the composition $\varepsilon_{I,\{L_i\},M}\left(id_{\mathcal{P}_I(\{L_i\},M)}, \varepsilon_{E_i,\{K_e\},L_i}\right)$ by the canonical isomorphism κ^{-1}. Note that if $E = \emptyset$ then $c = \varepsilon$.

One may check that the composition is associative in the obvious sense.

Example. Consider $L \in \mathcal{M}$ such that $\mathcal{E}nd\, L = \mathcal{H}om(L, L)$ exists. Then the composition $c \in P_2(\{\mathcal{E}nd\, L, \mathcal{E}nd\, L\}, \mathcal{E}nd\, L)$ is associative, hence it makes $\mathcal{E}nd\, L$ a monoid. In the k-linear setting $\mathcal{E}nd\, L$ is an associative k-algebra.

1.2.3 Example. Assume we are in situation 1.1.13, so $\mathcal{M} = \mathrm{mod}R$ and the pseudo-tensor structure is given by the functors P^* defined by the operad $\mathcal{B}$. For L_i, M as above set $X := \mathrm{Hom}_{R^{\otimes I}}(\underset{I}{\otimes} L_i, M \underset{R}{\otimes} \mathcal{B}_{\widetilde{I}})$. This is a right R-module in the obvious way, and we have the canonical operation $\varepsilon \in P^*_{\widetilde{I}}(\{L_i, X\}, M)$. The maps $\kappa_J(\varepsilon)$ are isomorphisms for $|J| = 1$, i.e., X necessarily coincides with $\mathcal{P}_I(\{L_i\}, M)$ if the latter object exists. This happens in the following situations:

(i) Assume that the composition maps $\mathcal{B}_I \underset{R^{\otimes I}}{\otimes} \mathcal{B}_{J/I} \to \mathcal{B}_J$ are isomorphisms. Then $X = \mathcal{P}_I(\{L_i\}, M)$ if L_i are projective modules of finite rank.

(ii) Assume in addition that all $\mathcal{B}_I$ are flat as R-modules. Then $X = \mathcal{P}_I(\{L_i\}, M)$ if L_i are finitely presented R-modules.

1.2.4 Augmented pseudo-tensor structures. Denote by $\mathcal{S}^{\wedge}$ the category of arbitrary finite sets and maps between them, so $\mathcal{S}$ is a subcategory of $\mathcal{S}^{\wedge}$. If we replace in 1.1.1 $\mathcal{S}$ by $\mathcal{S}^{\wedge}$ then we get the definition of *augmented pseudo-tensor category*. An augmented pseudo-tensor category with single object is called an *augmented operad*.

Examples. (i) A tensor category with a unit object $\mathbf{1}$ is an augmented tensor category: set $P_I(\{L_i\}, M) = \mathrm{Hom}(\underset{I}{\otimes} L_i, M)$. We assume that $\underset{\phi}{\otimes} = \mathbf{1}$.

(ii) The linear orders operad Σ (see 1.1.4) extends in the obvious way to an augmented operad $\Sigma^{\wedge}$ with $\Sigma^{\wedge}_{\emptyset} = \{\text{point}\}$. We also have the "trivial" augmented one-point operad, and their k-linear envelopes $\mathcal{A}ss^{\wedge} = k[\Sigma^{\wedge}]$ and $\mathcal{C}om^{\wedge} = k$.

1.2.5. One may consider an augmented pseudo-tensor category as a usual pseudo-tensor category equipped with some extra structure; let us describe it explicitly. We need the

following definition.

Let $\mathcal{M}$ be any pseudo-tensor category. Assume we have a functor

$$h\colon \mathcal{M} \to Sets, \tag{1.2.5.1}$$

and for any finite set I of order ≥ 2 and $i_0 \in I$ we have natural maps

$$h_{I,i_0}\colon P_I(\{L_i\}, M) \times h(L_{i_0}) \to P_{I \smallsetminus \{i_0\}}(\{L_i\}, M) \tag{1.2.5.2}$$

(here, as usually, $\{L_i\}$ is an I-family of objects, $M \in \mathcal{M}$). We call these data an *augmentation functor* if the following compatibilities hold:

(i) Let J be a finite set of order ≥ 2, $\pi\colon J \twoheadrightarrow I$ a surjective map, $j_0 \in J$. Then for $\varphi \in P_I(\{L_i\}, M)$, $\psi_i \in P_{J_i}(\{K_j\}, L_i)$, $a \in h(K_{j_0})$ one has

$$h_{J,j_0}(\varphi(\psi_i), a) = \varphi(\psi_{i'}, h_{J_{i_0},j_0}(\psi_{i_0}, a))$$

(here $i_0 = \pi(j_0), i' \in I \smallsetminus \{i_0\}$) if $|J_{i_0}| \geq 2$, and

$$h_{J,j_0}(\varphi(\psi_i), a) = h_{I,i_0}(\varphi, \psi_{i_0}(a))(\psi_{i'})$$

if $|J_{i_0}| = 1$.

(ii) Assume that $|I| \geq 2$; let $i_0, i_1 \in I$ be two distinct elements. Then for $\varphi \in P_I(\{L_i\}, M)$, $a_{i_0} \in h(L_{i_0})$, $a_{i_1} \in h(L_{i_1})$, one has

$$h_{I \smallsetminus \{i_0\}, i_1}(h_{I,i_0}(\varphi, a_0), a_1) = h_{I \smallsetminus \{i_1\}, i_0}(h_{I,i_1}(\varphi, a_1), a_0) \in P_{I \smallsetminus \{i_0,i_1\}}(\{L_i\}, M)$$

if $|I| > 2$, and

$$h_{I,i_0}(\varphi, a_0)a_1 = h_{I,i_1}(\varphi, a_1)a_0 \in h(M)$$

if $|I| = 2$.

Remark. Any subfunctor of an augmentation functor is itself an augmentation functor. The functor $h(M) \equiv \emptyset$ is an augmentation functor.

Definition. We say that our augmentation functor is *non-degenerate* if all the maps $P_I(\{L_i\}, M) \to \mathcal{M}aps(h(L_{i_0}), P_{I \smallsetminus \{i_0\}}(\{L_i\}, M))$ coming from 1.2.5.2 are injective.

Now let $\mathcal{M}^{\wedge}$ be an augmented pseudo-tensor category, $\mathcal{M}$ the corresponding usual pseudo-tensor category. Then $\mathcal{M}$ carries an augmentation functor $h = P_{\emptyset}$, h_{I,i_0} is the composition map for the embedding $I \smallsetminus \{i_0\} \hookrightarrow I$.

1.2.6 Lemma. The above construction yields $1-1$ correspondence between augmented pseudo-tensor categories and usual pseudo-tensor categories equipped with an augmentation functor. □

1.2.7. Note that any augmentation functor $h: \mathcal{M} \to Sets$ is automatically a pseudo-tensor functor (we consider $Sets$ as a tensor category with $\otimes = \Pi$), i.e., we have natural maps

$$h_I: P_I(\{L_i\}, M) \to \mathrm{Hom}(\underset{I}{\otimes} h(L_i), h(M))$$

To see this we may assume, by the above lemma, that h came from an augmented pseudo-tensor structure. Then h_I is the composition map for $\emptyset \to I$. Equivalently, for $\varphi \in P_n(\{L_i\}, M)$, $a_i \in h(L_i)$ one has (here $h_n := h_{\{1,\dots,n\}}$)

$$h_n(\varphi)(a_1 \otimes \cdots \otimes a_n) = h_{\{n-1,n\},n-1}(\dots(h_{\{1,\dots,n\},1}(\varphi, a_1)\dots), a_{n-1})a_n$$

In the situation 1.2.1 there is a canonical map

$$\bar{h}: h(\mathcal{P}_I(\{L_i\}, M)) \to P_I(\{L_i\}, M), \quad \lambda \longmapsto h_{\tilde{I},\cdot}(\varepsilon_{I,\{L_i\},M}, \lambda). \tag{1.2.7.1}$$

For c as in 1.2.2 the $\bar{h}$ arrows intertwine $h_{\tilde{I}}(c)$ and the usual composition map for P operations.

We say that $\mathcal{P}_I(\{L_i\}, M)$ is an *inner P object in the augmented sense* if (1.2.7.1) is an isomorphism. This amounts to the fact that the canonical morphism (1.2.2.1) (for $X = \mathcal{P}_I(\{L_i\}, M)$) is an isomorphism for $J = \emptyset$ as well.

1.2.8. One defines an *augmented pseudo-tensor functor* $\tau^\wedge: \mathcal{N}^\wedge \to \mathcal{M}^\wedge$ between augmented pseudo-tensor categories in the obvious way. Such $\tau^\wedge$ amounts to a usual pseudo-tensor functor $\tau: \mathcal{N} \to \mathcal{M}$ together with a morphism of augmentation functors $h^\wedge_{\mathcal{N}} \to h^\wedge_{\mathcal{M}}\tau$ that are compatible in the obvious manner. For example, given an augmented operad $\mathcal{B}^\wedge$ we have the notion of $\mathcal{B}^\wedge$ algebra in any augmented pseudo-tensor category $\mathcal{M}^\wedge$.

Examples. $\Sigma^\wedge$ algebras in $\mathcal{M}^\wedge$ are semi-groups; algebras for the trivial augmented operad are commutative semi-groups. In the k-linear setting $\mathcal{A}ss^\wedge$ algebras are unital associative

algebras, $\mathcal{C}om^\wedge$ algebras are unital commutative algebras. If A is a unital associative algebra then the image $1_A \in h(A)$ of $1 \in k = \mathcal{A}ss_\emptyset$ is called *the unit element* in $h(A)$.

Remark. 1_A is the unit of the associative algebra $h(A)$. So a unital associative algebra in $\mathcal{M}^\wedge$ is the same as an associative algebra in $\mathcal{M}$ such that the associative algebra $h(A)$ is unital and the left and right product with the unit element of $h(A)$ is identity morphism of A.

1.2.9. Following the pattern of 1.1.13, one may use augmented operads to define augmented pseudo-tensor structures on some categories of modules and their duals.

Let R be an associative k-algebra and $\mathcal{B}^\wedge$ an augmented strict R-operad, i.e., $\mathcal{B}$ is an augmented k-operad equipped with an isomoprhism $R \underset{\sim}{\longrightarrow} \mathcal{B}_1^\wedge$. Such $\mathcal{B}^\wedge$ is the same as a usual strict R-operad $\mathcal{B}$ together with a left R-module $h = \mathcal{B}_0$ and a system of $R^{\otimes I \smallsetminus \{i\}}$-linear morphisms $h_{I,i_0}: \mathcal{B}_I \underset{R}{\otimes} \mathcal{B}_0 \to \mathcal{B}_{I \smallsetminus \{i_0\}}$ (here $|I| \geq 2$, $i_0 \in I$, and $\mathcal{B}_I$ is considered as a right R-module via the i_0-th action) that satisfy obvious compatibilities.

According to 1.1.13 $\mathcal{B}$ defines the $*$ and $!$ pseudo-tensor structures on $\mathrm{mod}R$ and $R\mathrm{mod}^\circ$, respectively. Now h defines the augmentation functors h^*, $h^!$ for these structures. Namely, for $M \in \mathrm{mod}R$, $D \in R\mathrm{mod}$ set

$$h^*(M) := M \underset{R}{\otimes} h, \qquad h^!(D) := \mathrm{Hom}_{R\mathrm{mod}}(D, h)$$

The morphisms $h^*_{I,i_0}: P_I^{*\mathcal{B}}(\{L_i\}, M) \otimes h^*(L_{i_0}) \to P_{I \smallsetminus \{i_0\}}^{*\mathcal{B}}(\{L_i\}, M)$, $h^!_{I,i_0}: P_I^{!\mathcal{B}}(D, \{C_i\}) \otimes h^!(C_{i_0}) \to P^!_{I \smallsetminus \{i_0\}}(D, \{C_i\})$ come from h_{I,i_0} maps in the obvious manner. Compatibilities 1.2.5 (i), (ii) are clear. Note that $\mathcal{B}^\wedge = P^{*\wedge}(R) = P^{!\wedge}(R)$.

1.2.10 Example. Assume that R has a cocommutative Hopf algebra structure with counit $\varepsilon: R \to k$. It defines an augmented strict R-operad $\mathcal{B}^{R\,\wedge}$. Namely, we already defined $\mathcal{B}^R$ in 1.1.14. Set $h = k$ (the R-module structure is defined by ε), and let h_{I,i_0} be the obvious isomorphisms $R^{\otimes I} \underset{R}{\otimes} k = R^{\otimes I \smallsetminus \{i_0\}} \otimes (R \underset{R}{\otimes} k) \underset{\sim}{\longrightarrow} R^{\otimes I \smallsetminus \{i_0\}}$. We leave it to the reader to check the compatibilities. Note that k is a unit object for the standard tensor structure on Rmod, and $!$ augmented structure is the corresponding augmented pseudo-tensor structure (see example 1.2.4).

1.2.11 Modules over "operadic" algebras. Let $\mathcal{C}$ be a pseudo-tensor category which has two objects $\bullet$ and $*$, and $P_I^{\mathcal{C}}(\{L_i\}, M) = \emptyset$ whenever either ≥ 2 of L_i are equal to $*$, or exatly one of the objects $\{L_i, M\}$ is equal to $*$. In other words, the only (possibly) non-empty sets of operations are the operad $\mathcal{B} := P^{\mathcal{C}}(\bullet)$ (see 1.1.6 (iii)) and $\mathcal{C}_I := P_{\tilde{I}}^{\mathcal{C}}(\{\bullet_i, *\}, *)$, I is a finite set. We will say that $\mathcal{C}$ is a $\mathcal{B}$ *module operad*. We leave it to the reader to spell explicitly the definition of $\mathcal{B}$ module operad in terms of the composition data for $\mathcal{C}$- and $\mathcal{B}$-operations.

The above definition may be rewritten in the k-linear setting in the obvious way.

1.2.12 Examples. (i) Let $\mathcal{B}$ be an operad. Consider the pseudo-tensor category $\mathcal{B}'$ with two objects $\{\bullet, *\}$ which is equivalent to $\mathcal{B}$. Let $\mathcal{B}^m \subset \mathcal{B}'$ be the pseudo-tensor category with the same set of objects as $\mathcal{B}'$ and $P_I^{\mathcal{B}^m}(\{L_i\}, M) = P_I^{\mathcal{B}'}(\{L_i\}, M) = \mathcal{B}_I$ if either $L_i = M = \bullet$, or $M = \bullet$ and exactly one of L_i equals to $\bullet$; otherwise $P_I^{\mathcal{B}^m} = \emptyset$ (or 0 in the k-linear version). Then $\mathcal{B}^m$ is a $\mathcal{B}$ module operad, so $\mathcal{C}_I = \mathcal{B}_{I \sqcup \{*\}}$.

(ii) Consider the operad Σ (see 1.1.4). The Σ module operad Σ^m contains two Σ module suboperads Σ_ℓ^m, $\Sigma_r^m \subset \Sigma^m$. Namely, $\Sigma_I^m := \Sigma_{I \sqcup \{*\}}$ = set of linear orders on $I \sqcup \{*\}$, and $\Sigma_{\ell I}^m$ (resp. Σ_{rI}^m) is the subset of orders such that $*$ is the maximal (resp. the minimal) element.

(iii) Let $\mathcal{C}_i$ be $\mathcal{B}$ module operads, $i \in I$. Denote by $\prod_I^0 \mathcal{C}_i \subset \prod_I \mathcal{C}_i$ the full pseudo-tensor subcategory with objects $\bullet := (\bullet_i)$, $* := (*_i)$. This is a $\prod_I \mathcal{B}_i$ module operad.

1.2.13. Let $\mathcal{C}$ be a $\mathcal{B}$ module operad, $\mathcal{M}$ a pseudo-tensor category. Let $\tau\colon \mathcal{C} \to \mathcal{M}$ be a pseudo-tensor functor. Then $L := \tau(\bullet)$ is a $\mathcal{B}$ algebra in $\mathcal{M}$. We call $M := \tau(*)$ an *L-module with respect to* $\mathcal{C}$, or simply *L-module* (if $\mathcal{C}$ is fixed). Explicitly, a structure of L-module on an object $M \in \mathcal{M}$ is given by a system of compatible "action" maps $\mathcal{C}_I \to P_{\tilde{I}}(\{L_i, M\}, M)$.

Denote by $\mathcal{C}(\mathcal{M})$ the category of pseudo-tensor functors $\tau\colon \mathcal{C} \to \mathcal{M}$ (see 1.1.5); its objects are pairs (L, M), L is a $\mathcal{B}$ algebra, M is an L-module. If L is a fixed $\mathcal{B}$ algebra, then we denote by $\mathcal{M}(L) = \mathcal{M}(L, \mathcal{C}) \subset \mathcal{C}(\mathcal{M})$ the subcategory of L-modules: its objects are pairs (L, M), the morphisms are identity on L.

Example. Let L be an object of a pseudo-tensor category $\mathcal{M}$ such that $\mathcal{E}ndL \in \mathcal{M}$ exists.

Then L is an $\mathcal{E}nd\,L$-module with respect to Σ^m_ℓ.

The above definition may be rewritten in the k-linear setting in the obvious way. For a k-operad $\mathcal{B}$, a $\mathcal{B}$ module k-operad $\mathcal{C}$, and a $\mathcal{B}$ algebra L in a pseudo-tensor k-category $\mathcal{M}$ the category Lmod is a k-category.

1.2.14 Lemma. If $\mathcal{M}$ is an abelian pseudo-tensor category, then $\mathcal{M}(L)$ is an abelian k-category. $\square$

1.2.15 Examples. (i) $\mathcal{B} = \mathcal{L}ie$, $\mathcal{C} = \mathcal{B}^m$. Then $\mathcal{M}(L)$ is the usual category of modules over the Lie algebra L: its object is $M \in \mathcal{M}$ equipped with a pairing $\cdot \in P_2(\{L, M\}, M)$ that satisfies the condition $\ell_1 \cdot (\ell_2 \cdot m) - \ell_2 \cdot (\ell_1 \cdot m) = [\ell_1, \ell_2] \cdot m$.

(ii) $\mathcal{B} = \mathcal{A}ss = k[\Sigma]$. Let L be an associative algebra. If $\mathcal{C} = \mathcal{B}^m = k[\Sigma^m]$ then $\mathcal{M}(L, \mathcal{C})$ is the category of L-bimodules. If $\mathcal{C} = k[\Sigma^m_\ell]$ (resp. $\mathcal{C} = k[\Sigma^m_r]$) then $\mathcal{M}(L, \mathcal{C})$ is the category of left (resp. right) L-modules.

1.2.16. Let $\mathcal{B}$ be an operad, $\mathcal{C}$ a $\mathcal{B}$ module operad, and L a $\mathcal{B}$ algebra in the tensor category $Sets$ (with $\otimes = \Pi$, see 1.2.7). We have the obvious "forgetting of L-module structure" functor o: $Sets(L, \mathcal{C}) \to Sets$. Denote by $\mathcal{C}(L)$ the monoid of endormorphisms of this functor.

Here is an explicit description of $\mathcal{C}(L)$. It is generated by elements $((\ell_i), \varphi)$, I is a finite set, $\ell_i \in L$ is an I-family of elements of L, $\varphi \in \mathcal{C}_I$. The relations for $\mathcal{C}(L)$ come from surjective maps $\pi\colon \widetilde{J} \to \widetilde{I}$, $\pi(\cdot) = \cdot$, and collections $\psi_i \in \mathcal{B}_{J_i}$, $i \in I$. The relation corresponding to $\pi, (\psi_i)$ is

$$((\ell_j), \varphi_1(\varphi_2, \psi_i)) = ((\psi_i(\ell_j)_{j \in J_i}, \varphi_1) \cdot ((\ell_j)_{j \in J_\cdot}, \varphi_2)$$

Here (ℓ_j) is a J-family of elements of L, $\varphi_1 \in \mathcal{C}_I$, $\varphi_2 \in \mathcal{C}_{J_\cdot}$, where $J_\cdot := J \cap \pi^{-1}(\cdot)$. The element $1 = (\emptyset, id.)$ is the unit in $\mathcal{C}(L)$. Denote by $\mathcal{C}(L)$mod the category of sets equipped with $\mathcal{C}(L)$-action. The functor o lifts to the functor $\tilde{o}$: $Sets_{L\mathcal{C}} \to \mathcal{C}(L)$mod which is an equivalence of categories. More generally, one may define a $\mathcal{C}$ action of L on an object of any category, and this is the same as a $\mathcal{C}(L)$-action.

The above construction has its k-linear version. Namely, for a k-operad $\mathcal{B}$, a $\mathcal{B}$ module operad $\mathcal{C}$, and a $\mathcal{B}$ algebra L in $k$mod (or in the category of sheaves of k-modules on some space) we get an associative k-algebra with unit $\mathcal{C}(L)$ (or a sheaf of k-algebras) such that the category of L-modules coincides with the one of $\mathcal{C}(L)$-modules. We call $\mathcal{C}(L)$ the *enveloping algebra* of L. In situation 1.2.15 (i) it coincides with the usual enveloping algebra.

1.2.17. Let now $\mathcal{M}^{\wedge} = (\mathcal{M}, h)$ be an augmented pseudo-tensor category, $\mathcal{B}$, $\mathcal{C}$ as above. Let L be a $\mathcal{B}$ algebra in $\mathcal{M}$, and $M \in \mathcal{M}(L, \mathcal{C})$ an L-module. Then $h(L)$ is a $\mathcal{B}$ algebra in $Sets$ (since h is a pseudo-tensor functor), and M (considered as an object of the usual category $\mathcal{M}$) carries a canonical $\mathcal{C}(h(L))$-action. Explicitly, a generator $((\ell_i), \varphi) \in \mathcal{C}(h(L))$ acts as $h_I(a_I(\varphi))(\ell_i) \in \operatorname{End} M$. Therefore we have a canonical "identity" functor

$$\mathcal{M}(L, \mathcal{C}) \to \mathcal{C}(h(L))\text{-modules in } \mathcal{M}. \tag{1.2.17.1}$$

1.2.18 Example. Let $\mathcal{M}$ be a pseudo-tensor k-category, L a Lie algebra in $\mathcal{M}$. Let $\{M_i\}_{i\in I}$, N be a finite collection of L-modules. An operation $\varphi \in P_I(\{M_i\}, N)$ is *L-compatible* if $\cdot_N(id_L, \varphi) = \sum_{i_0 \in I} \varphi(\cdot_{M_{i_0}}, id_{M_i})_{i\neq i_0} \in P_{\tilde{I}}(\{L, M_i\}, N)$.[8] Let $P_{LI}(\{M_i\}, N) \subset P_I(\{M_i\}, N)$ be the k-submodule of L-compatible operations. It is easy to see that composition of L-linear operations is L-linear. Therefore P_{LI} is a pseudo-tensor structure on $\mathcal{M}(L)$. If $\mathcal{M}$ was an abelian pseudo-tensor category, then $\mathcal{M}(L)$ is also an abelian pseudo-tensor category (see 1.2.14). If h is a k-linear augmentation functor on $\mathcal{M}$ then $h(L)$ is a Lie k-algebra, and for every $M \in \mathcal{M}(L)$ $h(M)$ is an $h(L)$-module. The functor h^L of $h(L)$-invariants, $M \mapsto h^L(M) := h(M)^{h(L)}$, is an augmentation functor on $\mathcal{M}(L)$.

If $\{F_j\}_{j\in J}$, G are $h(L)$-modules in $\mathcal{M}$ then we define the subspace $P_J^{h(L)}(\{F_j\}, G) \subset P_J(\{F_j\}, G)$ of $h(L)$-linear operations in the obvious way. Therefore $h(L)$-modules form a pseudo-tensor k-category, and it is easy to see that the "identity" functor

$$\mathcal{M}(L) \to h(L)\text{-modules in } \mathcal{M} \tag{1.2.18.1}$$

[8] Here $\cdot_N \in P_2(\{L, N\}, N)$, $\cdot_{M_i} \in P_2(\{L, M_i\}, M_i)$ are the L-actions.

is a pseudo-tensor functor.

For a k-operad $\mathcal{B}$ a $\mathcal{B}$ algebra in $\mathcal{M}(L)$ is the same as a $\mathcal{B}$ algebra A in $\mathcal{M}$ on which L acts by derivations. According to (1.2.18.1) such A carries a canonical $h(L)$-action by derivations.

§3 Compound tensor categories

We recommend the reader to consider our basic example of compound tensor category (see §2 of ch.2) before, or while, reading this section. He may also restrict himself to compound tensor categories (see 1.3.12 - 1.3.15) instead of general compound pseudo-tensor categories (1.3.4 - 1.3.7), but in this case the duality 1.3.8(i) will be lost.

1.3.1. We have to fix some notation. For a finite non-empty set I denote by $Q(I)$ the set of all equivalence relations on I. We identify an element of $Q(I)$ with the corresponding quotient map $\pi_S: I \to S$ and denote it by (S, π_S) or simply S by abuse of notation; as usual for $s \in S$ we set $I_s := \pi_S^{-1}(s)$. The set $Q(I)$ is ordered: we write $S_1 \leq S_2$ iff S_1 is a quotient of S_2. It is a lattice: for any $S, T \in Q(I)$ there exist $\inf\{S,T\} \in Q(I)$ and $\sup\{S,T\} \in Q(I)$. For any $S \in Q(I)$ we have the obvious identifications $Q(I)_{\leq S} := \{S' \in Q(I): S' \leq S\} = Q(S)$, $Q(I)_{\geq S} := \{S'' \in Q(I): S'' \geq S\} = \prod_S Q(I_s)$.

We say that $S, T \in Q(I)$ are *transversal* if $|S| + |T| = |I| + |\inf\{S,T\}|$; S and T are called *complementary* if they are transveral and $|\inf\{S,T\}| = 1$.

1.3.2 Proposition. (i) Let V be a non-zero vector space. Let us assign to $S \in Q(I)$ the corresponding "diagonal" subspace $V^S \subset V^I$. Then S,T are transversal iff so are the subspaces V^S, $V^T \subset V^I$ (i.e., $V^S + V^T = V^I$), and S, T are complementary iff, in addition, $V^S \cap V^T = V^{\cdot} \subset V^I$.

(ii) Consider the graph G with vertices labeled by $S \sqcup T$ and edges labeled by I, such that an edge $i \in I$ connects $\pi_S(i)$ and $\pi_T(i)$. Then S and T are complementary (resp. transversal) iff G is a tree (resp. forest).

Proof. (i) Note that $|S| = \dim V^S / \dim V$, and for any S, T one has $V^{\inf\{S,T\}} = V^S \cap V^T$, $S = T$ iff $V^S = V^T$. Therefore $|S| + |T| \leq |\sup\{S,T\}| + |\inf\{S,T\}|$, and the

equality holds iff $V^{\sup\{S,T\}} = V^S + V^T$. Hence $|S| + |T| \leq |I| + |\inf\{S,T\}|$, and equality holds iff $V^I = V^S + V^T$.

(ii) Note that $|\inf\{S,T\}|$ coincides with the number of components of G, and compute the Euler characteristic of G. □

1.3.3 Corollary. a. Assume that $S, T \in Q(I)$ are transversal. Then

(i) for any $\alpha \in \inf\{S,T\}$ the equivalence relations induced by S, T on I_α are complementary;

(ii) for any $J \in Q(I)$ such that $J \geq S$ the pair $T, J \in Q(I)$ is transversal as well as $\inf\{T,J\}$, $S \in Q(J)$. The map $Q(I)_{\geq S} \to Q(T)$, $J \mapsto \inf\{T,J\}$ is injective.

b. Assume we have $S, P \in Q(I)$ such that $\inf\{S,P\} = \cdot$. Then $P = \inf\limits_{T \in \mathcal{T}} \{T\}$ where $\mathcal{T}$ is the set of all $T \in Q(I)$ such that T, S are complementary, and $T \geq P$. □

1.3.4. Let $\mathcal{M}$ be a category. For any functors $F: \mathcal{M} \to Sets$, $G: \mathcal{M}^\circ \to Sets$ we denote by $\langle F, G\rangle$ the quotient of the disjoint union of sets $F(X) \times G(X)$, $X \in \mathcal{M}$, modulo the relations $(\varphi(f), g) = (f, {}^t\varphi(g))$, where $f \in F(X)$, $g \in G(Y)$, $\varphi \in \mathrm{Hom}(X,Y)$ (and X, Y are objects in $\mathcal{M}$). We tacitly assume that $\langle F, G\rangle$ is a well-defined set; to assure this it suffices to know that isomorphism classes of objects in $\mathcal{M}$ form a set, or that either F or G is a representable functor (if, say, $F = \mathrm{Hom}(X_F, \cdot)$ then $\langle F, G\rangle = G(X_F)$).

If we are in the k-linear situation, i.e., $\mathcal{M}$ is a k-category and F, G are k-linear functors with values in k-modules, then we denote by $\langle F, G\rangle_k$ the k-module which is the quotient of $\bigoplus\limits_{X \in \mathcal{M}} F(X) \underset{k}{\otimes} G(X)$ modulo the relations $\varphi(f) \otimes g = f \otimes {}^t\varphi(g)$. We have an obvious map of sets $\langle F, G\rangle \to \langle F, G\rangle_k$ which is bijection if either F or G is representable.

1.3.5. Now assume that both categories $\mathcal{M}$ and $\mathcal{M}^\circ$ carry the pseudo-tensor structures given by functors P_I^* and $P_I^!$, respectively. It will be convenient to write the variables for $P_I^!$ in reverse order, i.e., as $P_I^!(M, \{L_i\})$ so that $P_\cdot^!(M, L) = \mathrm{Hom}_{\mathcal{M}}(M, L)$.

For any $I, J \in \mathcal{S}$ and I- and J-families of objects $\{L_i\}, \{M_j\}$, respectively, set

$$P_{I,J}^{*!}(\{L_i\}, \{M_j\}) := \left\langle P_I^*(\{L_i\}, \cdot), P_J^!(\cdot, \{M_j\}) \right\rangle$$

Our $P_{I,J}^{*!}$ is a functor; moreover, the $*$ and $!$ operations act on $P^{*!}$. Precisely, for any

surjective maps $H \twoheadrightarrow J$, $G \twoheadrightarrow I$ and families of objects $\{N_h\}$, $\{K_g\}$ we have the composition map

$$P^!_{H/J}(\{M_j\},\{N_h\}) \times P^{*!}_{I,J}(\{L_i\},\{M_j\}) \times P^*_{G/I}(\{K_g\},\{L_i\}) \to P^{*!}_{G,H}(\{K_g\},\{N_h\})$$

Note that $P^{*!}_{I,\cdot} = P^*_I$, $P^{*!}_{\cdot,J} = P^!_J$.

1.3.6. Assume that for any $I \in \mathbb{S}$ and a complementary pair $S, T \in Q(I)$ we have a natural map

$$\langle \quad \rangle^I_{S,T} : P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) \to P^{*!}_{T,S}(\{N_t\},\{M_s\})$$

$((\varphi_s),(\psi_t)) \mapsto \langle(\varphi_s),(\psi_t)\rangle^I_{S,T}$. Here $\{L_i\}$, $\{M_s\}$, $\{N_t\}$ are arbitrary families of objects parametrized by I, S, T, respectively, and "natural" means that $\langle \quad \rangle^I_{S,T}$ is functorial with respect to M_s and N_t variables. Note that $\langle \quad \rangle^I_{S,T}$ extends uniquely to the family of maps

$$\langle \quad \rangle^I_{S,T} : \prod_S P^{*!}_{I_s,H_s}(\{L_i\},\{B_h\}) \times \prod_T P^{*!}_{G_t,I_t}(\{A_g\},\{L_i\}) \to P^{*!}_{G,H}(\{A_g\},\{B_h\})$$

labeled by surjecitons $H \twoheadrightarrow S$, $G \twoheadrightarrow T$, in a way compatible with composition of $*$ and $!$ operations (acting on A_g and B_h, respectively).

1.3.7 Definition. The data $(P^*, P^!, \langle \quad \rangle)$ form a *compound pseudo-tensor structure* on $\mathcal{M}$ if the following compatibility axioms hold (here I, S, T, $\{L_i\}$, $\{M_s\}$, $\{N_t\}$ are as above):

(i) Take $H \in Q(I)$ such that $H \geq S$; set $V := \inf\{H, T\}$. Then for any H-family of objects $\{K_h\}$ the following diagram commutes:

$$\begin{array}{ccc} P^*_{H/S}(\{K_h\},\{M_s\}) \times P^*_{I/H}(\{L_i\},\{K_h\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) & \longrightarrow & P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) \\ \downarrow \prod_V \langle \quad \rangle^{I_v}_{H_v,T_v} & & \downarrow \langle \quad \rangle^I_{S,T} \\ P^*_{H/S}(\{K_h\},\{M_s\}) \times \prod_V P^{*!}_{T_v,H_v}(\{N_t\},\{K_h\}) & \xrightarrow{\langle \quad \rangle^H_{S,V}} & P^{*!}_{T,S}(\{N_t\},\{M_s\}) \end{array}$$

Here the upper horizontal arrow is the product of composition maps for $I_s \twoheadrightarrow H_s$.

(ii) Take $G \in Q(I)$ such that $G \geq T$; set $U := \inf\{S, G\}$. Then for any G-family of objects $\{R_g\}$ the following diagram commutes:

$$\begin{array}{ccc} P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/G}(\{R_g\},\{L_i\}) \times P^!_{G/T}(\{N_t\}\{R_g\}) & \longrightarrow & P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) \\ \Big\downarrow \prod_U <\quad>^{I_u}_{S_u,G_u} & & \Big\downarrow <\quad>^I_{S,T} \\ \prod_U P^{*!}_{G_u,S_u}(\{R_g\},\{M_s\}) \times P^!_{G/T}(\{N_t\},\{R_g\}) & \xrightarrow{<\quad>^G_{U,T}} & P^{*!}_{T,S}(\{N_t\},\{M_s\}) \end{array}$$

The upper arrow is the product of composition maps for $I_t \twoheadrightarrow G_t$.

(iii) If $S = I$, $T = .$, $M_i = L_i$, then $\langle (id_{L_i}), \psi \rangle^I_{I,.} = \psi$ for any $\psi \in P^!_I(N, \{L_i\})$

(iv) If $S = .$, $T = I$, $N_i = L_i$, then $\langle \varphi, (id_{L_i}) \rangle^I_{.,I} = \varphi$ for any $\varphi \in P^*_I(\{L_i\}, M)$ $\square$

We call a category $\mathcal{M}$ equipped with a compound pseudo-tensor structure a *compound pseudo-tensor category*. We usually denote it by $\mathcal{M}^{*!}$, while $\mathcal{M}^*$ and $\mathcal{M}^{o!}$ denote $\mathcal{M}$ and $\mathcal{M}^o$ considered as plain pseudo-tensor categories (with the operations P^* and $P^!$, respectively).

1.3.8 Remarks. (i) The axioms (i) – (ii) and (iii) – (iv) are mutually dual, therefore a compound pseudo-tensor structure on $\mathcal{M}$ amounts to that on $\mathcal{M}^\circ$.

(ii) Any *tensor* structure on $\mathcal{M}$ defines a compound pseudo-tensor structure: set $P^*_I(\{L_i\}, M) = \mathrm{Hom}(\underset{I}{\otimes} L_i, M)$, $P^!_I(N, \{L_i\}) = \mathrm{Hom}(N, \underset{I}{\otimes} L_i)$, and $\langle (\varphi_s), (\psi)_t \rangle^I_{S,T} = (\underset{S}{\otimes} \varphi_s) \circ (\underset{T}{\otimes} \psi_t)$ (note that $P^{*!}_{T,S}(\{N_t\},\{M_s\}) = \mathrm{Hom}(\underset{T}{\otimes} N_t, \underset{S}{\otimes} M_s)$).

1.3.9. A *compound pseudo-tensor functor* $\tau^{*!}: \mathcal{N}^{*!} \to \mathcal{M}^{*!}$ between the compound pseudo-tensor categories is a pair of pseudo-tensor functors $\tau^*: \mathcal{N}^* \to \mathcal{M}^*$, $\tau^!: \mathcal{N}^{o!} \to \mathcal{M}^{o!}$ that give rise to the same usual functor $\tau: \mathcal{N} \to \mathcal{M}$, such that the diagrams

$$\begin{array}{ccc} P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) & \xrightarrow{<\quad>^I_{S,T}} & P^{*!}_{T,S}(\{N_t\},\{M_s\}) \\ \Big\downarrow{\scriptstyle \tau^* \times \tau^!} & & \Big\downarrow{\scriptstyle \tau^{*!}} \\ P^*_{I/S}(\{\tau L_i\},\{\tau M_s\}) \times P^!_{I/T}(\{\tau N_t\},\{\tau L_i\}) & \xrightarrow{<\quad>^I_{S,T}} & P^{*!}_{T,S}(\{\tau N_t\},\{\tau M_s\}) \end{array}$$

commute. Here the arrow $\tau^{*!}$ is defined in the obvious way using τ^* and $\tau^!$.

1.3.10. Let us explain what an augmented compound pseudo-tensor category is. Let $\mathcal{M}^{*!}$ be a compound tensor category and h an augmentation functor on $\mathcal{M}^*$. We say that h is *compatible* with the compound structure, or h is a $*$-*augmentation functor* on $\mathcal{M}^{*!}$ if it satisfies the following condition. Let $S, T \in Q(I)$ be a complementary pair and $i_0 \in I$

an element such that $\pi_T^{-1}(t_0) = \{i_0\}$, $|\pi_S^{-1}(s_0)| \geq 2$ where $s_0 = \pi_S(i_0)$, $t_0 = \pi_T(i_0)$. Set $I^0 := I \smallsetminus \{i_0\}$, $T^0 := T \smallsetminus \{t_0\}$. We may consider S and T^0 as a complementary pair in $Q(I^0)$. Our condition is commutativity of the diagram

$$\begin{array}{ccc} P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) \times h(N_{t_0}) & \longrightarrow & P^*_{I^0/S}(\{L_i\},M_s) \times P^!_{I^0/T^0}(N_t,\{L_i\}) \\ \downarrow < \quad >^I_{S,T} & & \downarrow < \quad >^{I^0}_{S,T^0} \\ P^{*!}_{T,S}(\{N_t\},\{M_s\}) \times h(N_{t_0}) & \longrightarrow & P^{*!}_{T^0,S}(\{N_t\},\{M_s\}) \end{array}$$

Here the upper horizontal arrow sends $((\varphi_s),(\psi_t),a)$ to $((\varphi_s, h_{I^0_{s_0,i_0}}(\varphi_{s_0},\psi_{t_0}(a))),\psi_t)$, and the lower one is the obvious extension of h_{T,t_0} to $P^{*!}_{T,\cdot}$.

One spells the compatibility property of a !-augmentation functor in a similar (dual) way. We call a compound pseudo-tensor category endowed with both $*$- and !-augmentation functors an *augmented* compound pseudo-tensor category.

The above definitions have obvious k-linear versions (replace product of sets by tensor product of k-modules, etc.).

1.3.11 Duality. The augmentation functors may be used to relate the pseudo-tensor categories $\mathcal{M}^*$ and $\mathcal{M}^{o!}$. For example, consider a !-augmentation functor $h^!$ on $\mathcal{M}^{*!}$. Assume that $h^!$ is representable by an object $\mathbf{1}$, so $h^!(M) = \mathrm{Hom}_{\mathcal{M}}(M,\mathbf{1})$, and for any $M \in \mathcal{M}$ the object $M' := \mathcal{H}om_{\mathcal{M}^*}(M,\mathbf{1})$ exists (here $\mathcal{H}om_{\mathcal{M}^*}$ means inner Hom with respect to $*$ structure, see 1.2.1). Then *the functor* $\mathcal{M}^\circ \to \mathcal{M}$, $M \mapsto M'$, *admits a canonical pseudo-tensor extension*

$$\mathcal{M}^{\circ!} \to \mathcal{M}^* \tag{1.3.11.1}$$

We need to define the maps $P^!_I(M,\{L_i\}) \to P^*_I(\{L'_i\},M')$. Let I^1, I^2 be two copies of I. Set $J := I^1 \sqcup I^2$, $\widetilde{I} = I \sqcup \cdot$, and let $\pi_I\colon J \twoheadrightarrow I$, $\pi_{\widetilde{I}}\colon J \twoheadrightarrow \widetilde{I}$ be the projections defined by formulas $\pi_I(i^1) = \pi_I(i^2) = i$, $\pi_{\widetilde{I}}(i^1) = i$, $\pi_{\widetilde{I}}(i^2) = \cdot$; here $i \in I$, and $i^1 \in I^1$, $i^2 \in I^2$ are the copies of i. Then $I, \widetilde{I} \in Q(J)$ is a complementary pair (e.g., by 1.3.2 (ii)). Consider a J-family of objects $\{A_j\}$, $A_{i^1} = L'_i$, $A_{i^2} = L_i$, an I-family of objects $\{\mathbf{1}_i\}$ (copies of $\mathbf{1}$), and an $\widetilde{I}$-family of objects which are L_i for $i \in I \subset \widetilde{I}$ and the $\cdot$ one is M. We define the

desired map is the composition

$$P^!_I(M,\{L_i\}) \to P^{*!}_{\tilde{I},I}(\{L'_i,M\},\{\mathbf{1}_i\}) \to P^*_{\tilde{I}}(\{L'_i,M\},\mathbf{1}) \xrightarrow{\sim} P^*_I(\{L'_i\},M')$$

Here the first arrow sends $\psi \in P^!_I(M,\{L_i\})$ to $\left\langle (\kappa_i),(\psi,id_{L'_i})\right\rangle^J_{I,\tilde{I}}$ (here $\kappa_i \in P^*_2(\{A_{i^1},A_{i^2}\},\mathbf{1}_i) = P^*_2(\{L'_i,L_i\},\mathbf{1})$ is the canonical pairing), the second arrow is $\lambda \mapsto h^!_I(\lambda)(id^{\otimes I}_{\mathbf{1}})$ (for $h^!_I$ see 1.2.7; we extend it to $P^{*!}$ in the obvious manner), and the last arrow is the canonical isomorphism (see 1.2.1). We leave it to the reader to check that the above construction is compatible with composition of operations.

1.3.12. Assume that the ! pseudo-tensor structure is representable (see 1.1.3). Then $P^*_{I,J}(\{L_i\},\{M_j\}) = P^*_I(\{L_i\},\underset{J}{\otimes^!}M_j)$, and we may rewrite the $\langle\ \ \rangle^I_{S,T}$ data as a collection of canonical morphisms (*the compound tensor product maps*)

$$\otimes^I_{S,T}\colon P^*_{I/S}(\{L_i\},\{M_s\}) = \prod_S P^*_{I_s}(\{L_i\},M_s) \to P^*_T(\{\underset{I_t}{\otimes^!}L_i\},\underset{S}{\otimes^!}M_s), \tag{1.3.12.1}$$

$\otimes^I_{S,T} := \langle\cdot,(id_{\underset{I_t}{\otimes^!}L_i})\rangle^I_{S,T}$. The axioms (i), (ii) of 1.3.7 may be rewritten as commutativity of the following diagrams (here $I,S,T,\dots$ have the same meaning as in 1.3.2):

(i)
$$\begin{array}{ccc} P^*_{H/S}(\{K_h\},\{M_s\}) \times P^*_{I/H}(\{L_i\},\{K_h\}) & \longrightarrow & P^*_{I/S}(\{L_i\},\{M_s\}) \\ \Big\downarrow \otimes^H_{S,V} \times \prod\limits_V \otimes^{I_v}_{H_v,T_v} & & \Big\downarrow \otimes^I_{S,T} \\ P^*_V(\{\underset{H_v}{\otimes^!}K_h\},\underset{S}{\otimes^!}M_s) \times P^*_{T/V}(\{\underset{I_t}{\otimes^!}L_i\},\underset{H_v}{\otimes^!}K_h) & \longrightarrow & P^*_T(\{\underset{I_t}{\otimes^!}L_i\},\underset{S}{\otimes^!}M_s) \end{array}$$

(the horizontal arrows are composition maps)

(ii)
$$\begin{array}{ccc} P^*_{I/S}(\{L_i\},\{M_s\}) & \xrightarrow{\otimes^I_{S,T}} & P^*_T(\{\underset{I_t}{\otimes^!}L_i\},\underset{S}{\otimes^!}M_s) \\ \Big\downarrow \prod\limits_U \otimes^{I_u}_{S_u,G_u} & & \Big\downarrow \\ P^*_{G/U}(\{\underset{I_g}{\otimes^!}L_i\},\{\underset{S_u}{\otimes^!}M_s\}) & \longrightarrow & P^*_T(\{\underset{G_t}{\otimes^!}(\underset{I_g}{\otimes^!}L_i)\},\underset{S}{\otimes^!}M_s) \end{array}$$

Here the right vertical arrow is the $*$ composition map with compatibility morphisms $\underset{G_t}{\otimes^!}(\underset{I_g}{\otimes^!}L_i) \to \underset{I_t}{\otimes^!}L_i$; the lower horizontal arrow is the composition of $\otimes^G_{U,T}$ and the map of P^*_T's arising from $\underset{U}{\otimes^!}(\underset{S_u}{\otimes^!}M_s) \to \underset{S}{\otimes^!}M_s$.

1.3.13. The simplest non-trivial $\otimes^I_{S,T}$ operations are those with $|S| = 2$ (the binary compound tensor products). They look as follows. If $S = \{1,2\}$ then the projection $I \twoheadrightarrow S$ amounts to a non-trivial decomposition $I = I_1 \sqcup I_2$. The complementarity of T, S means that the only non-trivial T-equivalence class I_t has two elements $i_1 \in I_1$, $i_2 \in I_2$, i.e., T coincides with $I_1 \vee I_2 =$ the union of I_1, I_2 with i_1, i_2 identified. We call the map

$$\underset{i_1,i_2}{\otimes} = \otimes^{I_1,I_2}_{i_1,i_2} := \otimes^I_{S,T} \colon P^*_{I_1}(\{L_i\}, M_1) \otimes P^*_{I_2}(\{L_i\}, M_2) \\ \to P^*_{I_1 \vee I_2}(\{L_i, L_{i_1} \otimes^! L_{i_2}\}, M_1 \otimes^! M_2),$$

$\varphi_1, \varphi_2 \mapsto \varphi_1 \underset{i_1,i_2}{\otimes} \varphi_2$, the *binary tensor product at* i_1, i_2 (here the objects L_i under the $P^*_{I_1 \vee I_2}$ sign are those for $i \in I \smallsetminus \{i_1, i_2\}$).

1.3.14. Assume that $\mathcal{M}^{o!}$ is actually a tensor category; we call such $\mathcal{M}^{*!}$ a *compound tensor category*. One may consider $\otimes^!$ as a tensor product on $\mathcal{M}$ itself; denote this tensor category as $\mathcal{M}^!$ (so $\mathcal{M}^!$ is the tensor category dual to $\mathcal{M}^{o!}$).

If we are in the k-linear situation, then we call $\mathcal{M}^{*!}$ *abelian* if $\mathcal{M}$ is an abelian k-category, P^*_I are left exact functors, and $\otimes^!$ is right exact.

A *compound tensor functor* $\tau^{*!} \colon \mathcal{N}^{*!} \to \mathcal{M}^{*!}$ between the compound tensor categories is a compound pseudo-tensor functor such that all the canonical morphisms $\nu_I \colon \tau(\underset{I}{\otimes^!} N_i) \to \underset{I}{\otimes} \tau(N_i)$ from 1.1.6 (ii) are isomorphisms, i.e., $\tau^! \colon \mathcal{N}^! \to \mathcal{M}^!$ is a tensor functor. Such $\tau^{*!}$ amounts to a pair $(\tau^*, \tau^!)$ where $\tau^* \colon \mathcal{N}^* \to \mathcal{M}^*$, $\tau^! \colon \mathcal{N}^! \to \mathcal{M}^!$ are, respectively, the pseudo-tensor and tensor extensions of the same functor $\tau \colon \mathcal{N} \to \mathcal{M}$ which commute with $\otimes^I_{S,T}$ maps.

Remark. If $\mathcal{M}$ is a compound tensor k-categrory then the category of superobjects in $\mathcal{M}$ is a compound tensor category in the obvious way, as well as the categories of graded super objects and DG superobjects (see 1.1.16 for the terminology).

1.3.15. If $\mathcal{M}^{*!}$ is a compound tensor category, then the right vertical arrow in the diagram 1.3.12 (ii) is an isomorphism. Therefore all the morphisms $\otimes^I_{S,T}$ are completely determined by binary ones (use this diagram to compute $\otimes^I_{S,T}$ inductively).

The binary tensor products are functorial, commutative and associative. Let us spell these properties explicitly. Functoriality means that the binary products are compatible

with composition of $*$ operations in the obvious sense, and $\otimes_{i_1,i_2}^{I_1,I_2}$ coincides with the usual ! tensor product of morphisms if $I_1 = \{i_1\}$, $I_2 = \{i_2\}$. Commutativity means that $\otimes_{i_1,i_2}^{I_1,I_2} = \otimes_{i_2,i_1}^{I_2,I_1}$. To spell the associativity property assume that we have three sets I_1, I_2, I_3 and the elements $i_1 \in I_1$, $i_2^1, i_2^3 \in I_2$, $i_3 \in I_3$; denote by $I_1 \vee I_2 \vee I_3$ the disjoint union of I_j's modulo the relation $i_1 = i_2^1$, $i_2^3 = i_3$. Take $\varphi_j \in P^*_{I_j}(\{L_i\}, M_j)$. The associativity says that

$$(\varphi_1 \underset{i_1,i_2^1}{\otimes} \varphi_2) \underset{i_2^3,i_3}{\otimes} \varphi_3 = \varphi_1 \underset{i_1,i_2^1}{\otimes} (\varphi_2 \underset{i_2^3,i_3}{\otimes} \varphi_3)$$
$$\in P^*_{I_1 \vee I_2 \vee I_3}(\{L_i, L_{i_1} \otimes^! L_{i_2^1}, L_{i_2^3} \otimes^! L_{i_3}\}, M_1 \otimes^! M_2 \otimes^! M_3)$$

if $i_2^1 \neq i_2^3$, and

$$(\varphi_1 \underset{i_1,i_2}{\otimes} \varphi_2) \underset{i_2,i_3}{\otimes} \varphi_3 = \varphi_1 \underset{i_1,i_2}{\otimes} (\varphi_2 \underset{i_2,i_3}{\otimes} \varphi_3) = (\varphi_1 \underset{i_1,i_3}{\otimes} \varphi_3) \underset{i_1,i_2}{\otimes} \varphi_2$$
$$\in P^*_{I_1 \vee I_2 \vee I_3}(\{L_i, L_{i_1} \otimes^! L_{i_2} \otimes^! L_{i_3}\}, M_1 \otimes^! M_2 \otimes^! M_3)$$

if $i_2^1 = i_2^3 = i_2$.

It is easy to check that for a category $\mathcal{M}$ equipped with a tensor structure $\otimes^!$ and a pseudo-tensor structure P^* the data of functorial, commutative and associative binary tensor products amount to the whole data of compound tensor products, i.e., to a compound tensor structure.

Remark.[9] The obvious Action of the tensor category $\mathcal{M}^!$ on $\mathcal{M}$ extends canonically to an $\mathcal{M}^!$-Action on $\mathcal{M}^*$ defined by the maps $P^*_I(\{M_i\}, N) \to P^*_I(\{A_i \otimes^! M_i\}, (\otimes A_i) \otimes^! N)$, $\varphi \mapsto \varphi \otimes \{id_{A_i}\}$.

1.3.16. Let $\mathcal{M}^{*!}$ be a compound tensor category such that $\mathcal{M}^!$ has unit object $\mathbf{1}$ (as a tensor category). Then $h^!(M) := \mathrm{Hom}_{\mathcal{M}}(M, \mathbf{1})$ is a !-augmentation functor on $\mathcal{M}^{o!}$. We say that $\mathbf{1}$ is a *unit object* in $\mathcal{M}^{*!}$ if $h^!$ is compatible with the compound structure (see 1.3.10). Explicitly, this means that for any I-family of objects $\{L_i\}$ and an object M the binary tensor product at $i_0 \in I$

$$\mathrm{End}\,\mathbf{1} \times P^*_I(\{L_i\}, M) \to P^*_I(\{L_i, \mathbf{1} \otimes^! L_{i_0}\}_{i \neq i_0}, \mathbf{1} \otimes^! M) = P^*_I(\{L_i\}, M)$$

[9] We use terminology from 1.1.6(v).

sends $id_{\mathbf{1}} \times \varphi$ to φ for any $\varphi \in P^*_I(\{L_i\}, M)$.

Recall that in any tensor category $F = \operatorname{End} \mathbf{1}$ is an abelian semigroup (a commutative k-algebra in the k-linear situation) that acts canonically on any object M (the action $F \to \operatorname{End} M$ sends $f \in F$ to $f \otimes^! id_M \in \operatorname{End}(\mathbf{1} \otimes^! M) = \operatorname{End} M$). The above compatibility implies that the F-action on $P^*_I(\{L_i\}, M)$ through M coincides with the action through any L_i. If we are in the k-linear situation this means that P^*_I are canonically F-modules and $\mathcal{M}^{*!}$ is actually a compound tensor F-category.

We will call a compound tensor category $\mathcal{M}^{*!}$ which has a unit and is endowed with a $*$-augmentation functor h simply an *augmented compound tensor category*. We leave the definition of an *augmented compound tensor functor* between such categories to the reader.

1.3.17. Assume we are in the k-linear situation. We say that our unit object $\mathbf{1}$ is a *strong unit object* in $\mathcal{M}^{*!}$ if for any $I \in \mathbb{S}$, and $L_i, M \in \mathcal{M}$ one has $P^*_I(\{L_i, \mathbf{1}\}, M) = 0$.

1.3.18. A *compound operad* is the same as a compound pseudo-tensor category with single object. One may use compound operads to define compound tensor structures on some categories of modules (cf. 1.1.13). As in 1.1.13 we explain this in the k-linear setting.

Let R be an associative k-algebra, and $\mathcal{B}^{*!}$ a compound *strict* R-operad, i.e., a compound k-operad $\mathcal{B}^{*!}$ together with isomorphism of k-algebras $R \underset{\sim}{\longrightarrow} \mathcal{B}_1$. Explicitly, such $\mathcal{B}^{*!}$ is a pair $(\mathcal{B}^*, \mathcal{B}^!)$, where $\mathcal{B}^*$ is a strict R-operad, and $\mathcal{B}^!$ is a strict R^o-operad (R^o is the opposite algebra, so $\mathcal{B}^*_I$ is an $(R - R^{\otimes I})$-bimodule and $\mathcal{B}^!_I$ is an $(R^{\otimes I} - R)$-bimodule), together with the pairings

$$\langle \quad \rangle^I_{S,T} : (\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) \to \mathcal{B}^!_S \underset{R}{\otimes} \mathcal{B}^*_T \tag{1.3.18.1}$$

defined for any complementary pair $S, T \in Q(I)$. These maps are morphisms of $(R^{\otimes S} - R^{\otimes T})$-bimodules, and the axioms 1.3.7 just say that the diagram (here I, S, T, H, V are as

in 1.3.7(i))

$$\begin{array}{ccc}
(\underset{S}{\otimes}\mathcal{B}^*_{H_s})\underset{R^{\otimes H}}{\otimes}(\underset{H}{\otimes}\mathcal{B}^*_{I_h})\underset{R^{\otimes I}}{\otimes}(\underset{T}{\otimes}\mathcal{B}^!_{I_t}) & \longrightarrow & (\underset{S}{\otimes}\mathcal{B}^*_{I_3})\underset{R^{\otimes I}}{\otimes}(\underset{T}{\otimes}\mathcal{B}^!_{I_t}) \\
\downarrow & & \\
(\underset{S}{\otimes}\mathcal{B}^*_{H_S})\underset{R^{\otimes H}}{\otimes}(\underset{V}{\otimes}\mathcal{B}^!_{H_v})\underset{R^{\otimes V}}{\otimes}(\underset{V}{\otimes}\mathcal{B}^*_{T_v}) & & \downarrow \\
\downarrow & & \\
\mathcal{B}^!_S\underset{R}{\otimes}\mathcal{B}^*_V\underset{R^{\otimes V}}{\otimes}(\underset{V}{\otimes}\mathcal{B}^*_{T_v}) & \longrightarrow & \mathcal{B}^!_S\underset{R}{\otimes}\mathcal{B}^*_T
\end{array}$$

commutes as well as the corresponding "dual" diagram, and $\langle\quad\rangle^I_{S,T}$ coincides with $id_{\mathcal{B}^*_T}$, $id_{\mathcal{B}^!_S}$ if $|\,S\,|=1$, resp. $|\,T\,|=1$.

Now any such $\mathcal{B}^{*!}$ defines a compound pseudo-tensor structure on mod$R$. Namely, set (see 1.1.13)

$$P^*_I(\{L_i\},M)=\mathrm{Hom}_{\mathrm{modR}^{\otimes I}}(\underset{I}{\otimes}L_i,\ M\underset{R}{\otimes}\mathcal{B}^*_I)$$

$$P^!_I(N,\{L_i\})=\mathrm{Hom}_{\mathrm{modR}}\big(N,(\underset{I}{\otimes}L_i)\underset{R^{\otimes I}}{\otimes}\mathcal{B}^!_I\big)$$

The composition of operations comes from composition laws in $\mathcal{B}^*$ and $\mathcal{B}^!$. Note that the ! pseudo-tensor structure is representable (one has $\overset{!}{\underset{I}{\otimes}}L_i=(\underset{I}{\otimes}L_i)\underset{R^{\otimes I}}{\otimes}\mathcal{B}^!_I$), so we may use the format of 1.3.12.

The compound tensor product

$$\otimes^I_{S,T}\colon\underset{S}{\otimes}P^*_{I_s}(\{L_i\},M_s)\to P^*_T(\{\overset{!}{\underset{I_t}{\otimes}}L_i\},\overset{!}{\underset{S}{\otimes}}M_s)$$

sends $\otimes\varphi_s\in\underset{S}{\otimes}\mathrm{Hom}_{\mathrm{modR}^{\otimes I_s}}(\underset{I_s}{\otimes}L_i,M_s\underset{R}{\otimes}\mathcal{B}^*_{I_s})$ to the composition

$$\begin{aligned}\underset{T}{\otimes}\big((\underset{I_t}{\otimes}L_i)\underset{R^{\otimes I_t}}{\otimes}\mathcal{B}^!_{I_t}\big)=(\underset{I}{\otimes}L_i)\underset{R^{\otimes I}}{\otimes}(\underset{T}{\otimes}\mathcal{B}^!_{I_t})&\xrightarrow{\otimes\varphi_s}(\underset{S}{\otimes}M_s)\underset{R^{\otimes S}}{\otimes}(\underset{S}{\otimes}\mathcal{B}^*_{I_s})\underset{R^{\otimes I}}{\otimes}(\underset{T}{\otimes}\mathcal{B}^!_{I_t})\\&\xrightarrow{\langle\quad\rangle^I_{S,T}}(\underset{S}{\otimes}M_s)\underset{R^{\otimes S}}{\otimes}\mathcal{B}^!_S\underset{R}{\otimes}\mathcal{B}^*_T\end{aligned}$$

We leave it to the reader to check commutativity of diagrams 1.3.12(i), (ii).

Note that $\otimes^!$ is a tensor product (i.e., we have a compound tensor structure) iff for any $J\twoheadrightarrow I$ the composition map $(\underset{I}{\otimes}\mathcal{B}^!_{J_i})\underset{R^{\otimes I}}{\otimes}\mathcal{B}^!_I\to\mathcal{B}^!_J$ is an isomorphism.

1.3.19. Now assume that $\mathcal{B}^{*!}$ is a $*$-augmented compound operad (see 1.3.10). So we have a left R-module $\mathcal{B}^*_0$ and the maps $\mathcal{B}^*_I \underset{R}{\otimes} \mathcal{B}^*_0 \to \mathcal{B}_{I\setminus\{i_0\}}, i_0 \in I$, that are compatible with $\langle\ \ \rangle$ in the sense explained in 1.3.10. According to 1.2.9, we get an augmentation functor h^* for the $*$ structure on $\mathrm{mod}R$, $h^*(M) := M \underset{R}{\otimes} \mathcal{B}^*_0$. We leave it to the reader to check that h^* is a $*$-augmentation functor for our compound pseudo-tensor structure (see 1.3.10). Similarly, a !-augmentation for $\mathcal{B}^{*!}$ defines a !-augmentation $h^!$ on $\mathrm{mod}R$ by formula $h^!(M) = \mathrm{Hom}_{\mathrm{mod}R}(M, \mathcal{B}^!_0)$.

1.3.20 Example. Assume we have a cocommutative Hopf algebra structure on R. It yields the similar structure on R^o (the coproduct map for R^o coincides with that for R). By 1.1.14 we have the strict R- and R^o operads $\mathcal{B}^* = \mathcal{B}^R$ and $\mathcal{B}^! = \mathcal{B}^{R^o}$ respectively. Let us show that $\mathcal{B}^*$ and $\mathcal{B}^!$ form a compound strict R-operad. We need to define the maps $\langle\ \ \rangle^I_{S,T}$. For $S \in Q(I)$ denote by $\triangle^I_S : R^{\otimes S} \to R^{\otimes I}$ the morphism of k-algebras $\underset{S}{\otimes} \triangle^{(I_s)}$ (recall that $\triangle^{(I_s)} : R \to R^{\otimes I_s}$ is the I_s-multiple coproduct). Let us consider $R^{\otimes I}$ as $(R^{\otimes S} - R^{\otimes T})$-bimodule with respect to the action $(\otimes r_s)(\otimes r_i)(\otimes r_t) := \triangle^I_S(\otimes r_s) \cdot (\otimes r_i) \cdot \triangle^I_T(\otimes r_t)$. We have the obvious indentifications of $(R^{\otimes S} - R^{\otimes T})$-bimodules $(\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) = (\underset{S}{\otimes} R^{\otimes I_s}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes} R^{\otimes I_t}) = (R^{\otimes I}) \underset{R^{\otimes I}}{\otimes} (R^{\otimes I}) = R^{\otimes I}$, $\mathcal{B}^!_S \underset{R}{\otimes} \mathcal{B}^*_T = R^{\otimes S} \underset{R}{\otimes} R^{\otimes T}$. Consider the morphism of $(R^{\otimes S} - R^{\otimes T})$-bimodules

$$\nu : R^{\otimes S} \underset{R}{\otimes} R^{\otimes T} \to R^{\otimes I},$$

$\nu\big((\otimes r_s) \otimes (\otimes r_t)\big) := (\otimes r_s)1(\otimes r_t) = \triangle^I_S(\otimes r_s) \cdot \triangle^I_T(\otimes r_t)$. If S, T are complementary then ν is an isomorphism $\big($use 1.3.2 (ii)$\big)$, and our $\langle\ \ \rangle^I_{S,T}$ is ν^{-1}.

We leave it to the reader to check that $\mathcal{B}^{*!} = (\mathcal{B}^*, \mathcal{B}^!, \langle\ \ \rangle)$ is actually a compound operad (i.e., that the diagram from 1.3.17 commutes). If our Hopf algebra has counit then the operads $\mathcal{B}^*$ and $\mathcal{B}^!$ are augmented in a natural way (see 1.2.10); it is easy to see that $(\mathcal{B}^{*\wedge}, \mathcal{B}^{!\wedge}, \langle\ \ \rangle)$ is an augmented compound operad.

§4 Rudiments of compound geometry

In this section $\mathcal{M}^{*!}$ is an abelian augmented compound tensor k-category (see 1.3.14,

1.3.16). We abbreviate $\otimes^!$ to $\otimes$. There are no assumptions about exactness of the $*$-augmentation functor h.

1.4.1. For $M \in \mathcal{M}$ set $\Gamma(M) := \mathrm{Hom}(\mathbf{1}, M)$. Then Γ is an augmentation functor on $\mathcal{M}^!$; in particular $\Gamma\colon \mathcal{M}^! \to k\mathrm{mod}$ is a pseudo-tensor functor (see 1.2.4, 1.2.7).

For a k-operad $\mathcal{B}$ we denote by $\mathcal{B}(\mathcal{M}^*)$ the category of $\mathcal{B}$ algebras in $\mathcal{M}^*$; we call them simply $\mathcal{B}^*$ *algebras*. In particular we have the category $\mathcal{L}ie(\mathcal{M}^*)$ of Lie* algebras. Similarly, if $\mathcal{B}^\wedge$ is an augmented k-operad then $\mathcal{B}^\wedge$ algebras in the augmented category $\mathcal{M}^*$ are called $\mathcal{B}^{\wedge *}$ algebras. In particular, we have the category of unital associative* algebras (see 1.2.8).

1.4.2 Matrix algebras. Assume we have $V, V' \in \mathcal{M}$ and $\langle\ \ \rangle \in P_2^*(\{V', V\}, \mathbf{1})$ a $*$ pairing. Then $E = V \otimes V'$ carries a canonical structure of associative* algebra which acts on V from the left and on V' from the right. Namely, the product operation $\cdot \in P_2^*(\{E, E\}, E)$ is the image of $id_V \otimes \langle\ \ \rangle \otimes id_{V'} \in \mathrm{E}nd(V) \otimes P_2^*(\{V', V\}, \mathbf{1}) \otimes \mathrm{E}nd(V')$ by the $\otimes^I_{S,T}$ map, where $I = \{1, 2, 3, 4\}$, and S, T are equivalence relations $\{2 = 3\}$ and $\{1 = 2, 3 = 4\}$ respectively. The E-action on V is the image of $id_V \otimes \langle\ \ \rangle$ by the binary map $\otimes^{\{1\},\{2,3\}}_{1,2}\colon \mathrm{E}nd(V) \otimes P_2^*(\{V', V\}, \mathbf{1}) \to P_2^*(\{E, V\}, V)$ (see 1.3.13 for notations). We leave the definition of the right E-action on V', as well as verification of associativity, to the reader.

One often wants to know if the E-action on V identifies E with $\mathcal{E}ndV$. The following condition on $\langle\ \ \rangle$ assures this:

Definition. A $*$ pairing $\langle\ \ \rangle \in P_2^*(\{V', V\}, \mathbf{1})$ is *non-degenerate* if for every finite set I and $A_{i\in I}, B \in \mathcal{M}$ the canonical map

$$P_I^*(\{A_i\}, B \otimes V') \to P_I^*(\{A_i, V\}, B), \quad \varphi \mapsto (id_B \otimes \langle\ \ \rangle)(\varphi, id_V), \tag{1.4.2.1}$$

is an isomorphism.[10]

If $\langle\ \ \rangle$ is non-degenerate then for every $B \in \mathcal{M}$ one has $B \otimes V' \xrightarrow{\sim} \mathcal{H}om(V, B)$ (see 1.2.1, 1.2.2) and[11] $h(B \otimes V') \xrightarrow{\sim} \mathrm{Hom}(V, B)$. In particular,[12] $V' = \mathcal{H}om(V, \mathbf{1})$, i.e., V' is

[10] Here $id_B \otimes \langle\ \ \rangle \in P_2^*(\{B \otimes V', V\}, B)$.
[11] Take $I = \emptyset$ in (1.4.2.1).
[12] Take $B = \mathbf{1}$.

the $*$ dual to V, and $E = V \otimes V' \xrightarrow{\sim} \mathcal{E}ndV$, $h(E) \xrightarrow{\sim} \mathrm{End}V$. Our E is a unital associative* algebra: the unit element[13] is $id_V \in \mathrm{End}V = h(E)$.

1.4.3. We will often use the fact that $\otimes$ behaves naturally with respect to $*$ actions on multiples. Precisely, assume we have a map $I \to J$ of finite sets and families of objects $\{L_i\}_{i\in I}$, $\{M_j\}_{j\in J}$, $\{M'_j\}_{j\in J}$. Then comes the canonical map

$$\otimes^{I\sqcup J}_{J,\tilde{I}}\colon \underset{J}{\otimes} P^*_{\tilde{I}_j}(\{L_i, M_j\}, M'_j) \to P^*_{\tilde{I}}(\{L_i, \underset{J}{\otimes} M_j\}, \underset{J}{\otimes} M'_j)\ ,$$

$\otimes\varphi_j \mapsto \underset{J}{\otimes} \varphi_j$, compatible with composition of operations.

Example. Assume we have a family $\{\mathcal{B}_j\}_{j\in J}$ of k-operads together with the corresponding $\mathcal{B}_j$ module operads $\mathcal{C}_j$. Let L_j be a $\mathcal{B}^*_j$ algebra, M_j an L_j-module in $\mathcal{M}^*$ with respect to $\mathcal{C}_j$. Then $\underset{J}{\otimes} M_j$ carries canonical L_j-actions that mutually commute. Namely, for $j_0 \in J$ the j_0's action map $\mathcal{C}_{j_0 I} \to P^*_{\tilde{I}}(\{L_{j_0 I}, \underset{J}{\otimes} M_j\}, \underset{J}{\otimes} M_j)$ sends $c \in \mathcal{C}_{j_0 I}$ to $c_{M_{j_0}} \otimes (\underset{j\neq j_0}{\otimes} id_{M_j})$, where $c_{M_{j_0}} \in P^*_{\tilde{I}}(\{L_{j_0 I}, M_{j_0}\}, M_{j_0})$ is the c-action on M_{j_0}.

Recall that $h(L_j)$ is a $\mathcal{B}_j$ algebra, and M_j is automatically an $h(L_j)$-module (see 1.2.17). It is easy to see that the action of $h(L_{j_0})$ on $\underset{J}{\otimes} M_j$ that comes from the above L_{j_0}-action on $\underset{J}{\otimes} M_j$ coincides with the action $a \mapsto a_{M_{j_0}} \otimes (\otimes id_{M_j})_{j\neq j_0}$.

1.4.4 Lie* algebras and modules. Let L be a Lie* algebra. Denote by $\mathcal{M}(L)$ the category of L-modules in $\mathcal{M}^*$. Let us show that

$$\mathcal{M}(L) \text{ is an } \textit{abelian augmented compound tensor } k\textit{-category}$$

in a natural way. We will denote it as $\mathcal{M}(L)^{*!}$.

According to 1.2.18 $\mathcal{M}(L)$ carries a canonical abelian pseudo-tensor structure P^*_L. If $\{M_j\}_{j\in J}$ is a finite family of L-modules then L acts on $\underset{J}{\otimes} M_j$ via the sum of the above J commuting actions (i.e., by Leibnitz rule). This is the ! tensor product of L-modules. One easily checks that $\otimes^I_{S,T}$ maps send L-linear operations to L-linear ones; this provides the maps $\otimes^I_{S,T}$ for $\mathcal{M}(L)$. Compatibility axioms 1.3.12 are clear.

[13] See 1.2.8.

The object **1** with trivial L-action is a unit object in $\mathcal{M}_L^{*!}$. For an L-module M the L-action on M yields the morphism of k-modules $h(M) \to \mathrm{Hom}_{\mathcal{M}}(L, M)$. Denote by $h^L(M)$ its kernel. h^L is a $*$-augmentation functor on $\mathcal{M}(L)^{*!}$.

Remark. A morphism $L' \to L$ of Lie* algebras yields the "restriction of action" augmented compound tensor functor $\mathcal{M}(L)^{*!} \to \mathcal{M}(L')^{*!}$.

1.4.5 The Chevalley complex I. The constructions below (except Remark (ii)) use only $*$ pseudo-tensor structure on $\mathcal{M}$.

For a Lie* algebra L and an L-module M we have *the Chevalley complex* $C(L, M)$ wich is a complex with terms $C^n(L, M) := P_n^*(\{L, .., L\}, M)_{sgn}$ (the skew-coinvariants of the action of the symmetric group Σ_n on operations of n variables) and differential given by the standard formula (see below). Set $H^\cdot(L, M) = H^\cdot C(L, M)$; these are *cohomology of* L *with coefficients in* M.

As in the case of usual Lie algebras, the Chevalley complex is naturally defined in the more general "super" setting[14] (below we often drop prefix "super"). Namely, let L be a super Lie* algebra, M a super L-module. Consider the graded super k-module $\oplus P_n^*(\{L[1], .., L[1]\}, M)_{\Sigma_n}$ (coinvariants of the symmetric group action on super operations of n variables). It carries a canonical odd differential δ of degree 1 coming from the morphism $P_n^*(\{L[1], .., L[1]\}, M)[1] \to P_n^*(\{L[1], .., L[1]\}, M)$, $\varphi \mapsto \cdot_M(id_{L[1]}, \varphi) - \frac{1}{2}\varphi([\ \ ], id_{L[1]}, .., id_{L[1]})$ where $\cdot_M \in P_2(\{L, M\}, M)$ is the action of L on M. One has $\delta^2 = 0$, so we have a DG super k-module denoted by $C(L, M)$. Let us describe its natural symmetries.

Set $L_\dagger := \mathcal{C}one(id_L)$. This complex carries a canonical structure of DG Lie* algebra: the bracket is uniqely determined by property that $L \hookrightarrow L_\dagger$ is a morphism of DG Lie* algebras (then the complementary graded submodule $L[1] \subset L_\dagger$ is a commutative ideal, and the adjoint action of $L \subset L_\dagger$ on it equals the adjoint action of L).

The DG Lie algebra $h(L)_\dagger$ acts on $C(L, M)$ in a canonical way. Namely, $h(L) \subset h(L_\dagger)$ acts via the adjoint action on copies of $L[1]$ and the L-module structure on M, and the

[14] See 1.1.16 for notation.

action of its complement $h(L)[1]$ comes from the map $h(L[1]) \otimes P_n^*(\{L[1], ..L[1]\}, M) \to P_{n-1}^*(\{L, .., L\}, M)$ which is minus sum of the canonical "convolution" maps for each of the n arguments.

Remarks. (i) If for every n the inner P object $\mathcal{P}_n^*(\{L, .., L\}, M)$ is well-defined (see 1.2.1) then one has the inner Chevalley complex $\mathcal{C}(L, M)$ defined in the obvious way. It carries a canonical $*$ action of $L_\dagger$. There is a canonical morphism of DG $h(L_\dagger)$-modules $h(\mathcal{C}(L, M)) \to C(L, M)$ which is an isomorphism if our $\mathcal{P}$'s are P objects in the augmented sense (see 1.2.7).

(ii) Let us assume that the $*$ dual $L' := \mathcal{H}om^*(L, \mathbf{1})$ is well-defined and the canonical pairing $\langle \quad \rangle \in P_2^*(\{L', L\}, \mathbf{1})$ is non-degenerate (see 1.4.2). Then for every $n \geq 0$ and every $N \in \mathcal{M}$ one has $N \otimes L'^{\otimes n} = \mathcal{P}_n^*(\{L, .., L\}, N)$ and $h(N \otimes L'^{\otimes n}) \xrightarrow{\sim} P_n^*(\{L, .., L\}, N)$ (see 1.2.1, 1.2.7). Thus the inner Chevalley complex is well-defined (for an L-module M one has $\mathcal{C}(L, M)^n = M \otimes \mathrm{Sym}^n(L'[-1])$), and we have a canonical isomorphism of complexes

$$h(\mathcal{C}(L, M)) \xrightarrow{\sim} C(L, M). \tag{1.4.5.1}$$

(iii) The above constructions are functorial in the obvious manner. In particular, if both L and M are equipped with a DG structure (i.e., L is a DG Lie* superalgbera, M is a DG super L-module) then the super k-module $C(L, M)$ acquires an extra DG structure. *The Chevalley complex of DG Lie* algebra L with coefficients in DG L-module M* is this super k-module equipped with the "total" grading and differential. Same for the inner Chevalley complex.

1.4.6 Commutative$^!$ algebras and modules. We can do some commutative algebra in $\mathcal{M}^!$ as in any tensor k-category. Denote by $\mathcal{C}omu(\mathcal{M}^!)$ the category of commutative and associative algebras with unit in $\mathcal{M}^!$. An object $A \in \mathcal{C}omu(M^!)$ (we call it a *commutative$^!$ algebra*) is an object $A \in \mathcal{M}$ together with a commutative and associative product $\cdot = \cdot_A \colon A \otimes A \to A$ and a morphism $1 = 1_A \colon \mathbf{1} \to A$ which is a unit for $\cdot$. For a finite family $\{A_i\}$ of commutative$^!$ algebras the tensor product $\underset{I}{\otimes} A_i$ has an obvious structure of commutative$^!$ algebra; this is the coproduct of $\{A_i\}$ in $\mathcal{C}omu(\mathcal{M}^!)$. Therefore $\otimes$

defines a tensor category structure on $\mathcal{C}omu(\mathcal{M}^!)$; the object $\mathbf{1}$ endowed with an obvious product is a unit in $\mathcal{C}omu(\mathcal{M}^!)$.

For $A \in \mathcal{C}omu(\mathcal{M}^!)$ we denote by $\mathcal{M}(A)$ the category of unital A-modules in $\mathcal{M}^!$; this is an abelian k-category. The usual tensor product $\underset{A}{\otimes}$ of A-modules is right exact, so $\mathcal{M}(A)$ is an abelian tensor k-category with unit object A. The "forgetting of A-action" functor $\mathcal{M}(A) \to \mathcal{M}$ admits a left adjoint "induction" functor $\mathcal{M} \to \mathcal{M}(A)$, $M \mapsto A \otimes M$, which is a tensor functor. Below for an A-module N we denote by $\cdot_N\colon A \otimes N \to N$ action map.

Let us show that

$$\mathcal{M}(A) \textit{ is an abelian augmented compound tensor } k\textit{-category}$$

in a natural way. We denote it $\mathcal{M}(A)^{*!}$.

The tensor structure has been discussed already, so it remains to define the $*$ operations P^*_{AI}, the compound tensor product maps $\otimes^I_{S,T}$, and the $*$-augmentation functor.

Take $I \in \mathbb{S}$ and objects $\{L_i\}_{i\in I}$, M of $\mathcal{M}$. Assume that some L_{i_0}, $i_0 \in I$, and M are A-modules. We say that an operation $\varphi \in P^*_I(\{L_i\}, M)$ is *A-linear with respect to the i_0th variable* if the operations $\cdot_M(\varphi \otimes_{i_0} id_A), \varphi(id_{L_i}, \cdot_{L_{i_0}})_{i\neq i_0} \in P^*_I(\{L_i, A \otimes L_{i_0}\}_{i\neq i_0}, M)$ coincide. Here $\varphi \otimes_{i_0} id_A \in P^*_I(\{L_i, A \otimes L_{i_0}\}_{i\neq i_0}, A \otimes M)$. If all L_i are A-modules, then we denote by $P^*_{AI}(\{L_i\}, M) \subset P^*_I(\{L_i\}, M)$ the k-submodule of *A-polylinear* (i.e., A-linear with respect to each variable) operations. The composition of A-polylinear operations is again A-polylinear, so P^*_{AI} form a pseudo-tensor structure on $\mathcal{M}(A)$. The functors P^*_{AI} are left exact.

Let us define the structure maps $\otimes^I_{S,T}\colon \underset{S}{\otimes} P^*_{AI_s}(\{L_i\}, M_s) \to P^*_{AT}(\{\underset{AI_t}{\otimes} L_i\}, \underset{AS}{\otimes} M_s)$. The projections $\underset{I_t}{\otimes} L_i \to \underset{AI_t}{\otimes} L_i$ define the embedding $P^*_{AT}(\{\underset{AI_t}{\otimes} L_i\}, \underset{AS}{\otimes} M_s) \hookrightarrow P^*_T(\{\underset{I_t}{\otimes} L_i\}, \underset{AS}{\otimes} M_s)$, and it is easy to see that the composition

$$\underset{S}{\otimes} P^*_{AI}(\{L_{i_s}\}, M_s) \to \underset{S}{\otimes} P^*_{I_s}(\{L_i\}, M_s) \xrightarrow{\otimes^I_{S,T}} P^*_T(\{\underset{I_t}{\otimes} L_i\}, \underset{S}{\otimes} M_s) \to P^*_T(\{\underset{I_t}{\otimes} L_i\}, \underset{AS}{\otimes} M_s)$$

takes values in the image of this embedding. Therefore we may consider the composition as a morphism $\underset{S}{\otimes} P^*_{AI_s}(\{L_i\}, M_s) \to P^*_T(\{\underset{AI_t}{\otimes} L_{i_t}\}, \underset{AS}{\otimes} M_s)$; this is $\otimes^I_{S,T}$ map for $\mathcal{M}(A)^{*!}$.

Compatibility axioms 1.3.12 for $\mathcal{M}(A)^{*!}$ follow from those for $\mathcal{M}^{*!}$.

The object A, considered as an A-module, is a unit object in $\mathcal{M}(A)^{*!}$. Note that the morphism $1\colon \mathbf{1} \to A$ yields a canonical isomorphism $End_{\mathcal{M}(A)}(A) \xrightarrow{\sim} \Gamma(A)$ of commutative k-algebras. Therefore, by 1.3.16, $\mathcal{M}(A)^{*!}$ is actually a compound tensor $\Gamma(A)$-category.

We define the $*$-augmentation functor on $\mathcal{M}_A^{*!}$ simply as restriction of h to $\mathcal{M}(A)$ (note that h_{I,i_0} maps preserve A-polylinear operations).

Remarks. (i) The "induction" functor $\mathcal{M} \to \mathcal{M}(A)$ extends naturally to an augmented compound tensor functor $\mathcal{M}^{*!} \to \mathcal{M}(A)^{*!}$. Namely, we already mentioned that the induction is a ! tensor functor. The map

$$P_I^*(\{F_i\}, G) \to P_{AI}^*(\{A \otimes F_i\}, A \otimes G) \tag{1.4.6.1}$$

is the composition $P_I^*(\{F_i\}, G) \to P_I^*(\{A \otimes F_i\}, A^{\otimes I} \otimes G) \to P_{AI}^*(\{A \otimes F_i\}, A \otimes G)$ where the first arrow sends φ to $\otimes_{\tilde{I},I}^{I \sqcup I}(\varphi, \{id_A\}_{i \in I})$, the second one is the composition with $\cdot_I \otimes id_G$ where $\cdot_I\colon A^{\otimes I} \to A$ is the I-fold product. We leave it to the reader to check the compatibility with $\otimes_{S,T}^I$ maps (see 1.3.14). The arrow $h(F) \to h(A \otimes F)$ is $h(1_A \otimes id_F)$.

More generally, any morphism of commutative$^!$ algebras $f : B \to A$ yields a compound tensor functor (base change)

$$f^* : \mathcal{M}(B)^{*!} \to \mathcal{M}(A)^{*!} \tag{1.4.6.2}$$

such that for $M \in \mathcal{M}(B)$ one has $f^*M = A \underset{B}{\otimes} M$. To define f^* you just replace in the above considerations $\mathcal{M}^{*!}$ by $\mathcal{M}(B)^{*!}$. The base change is compatible with composition of f's in the obvious way.

(ii) The induction pseudo-tensor functor $\mathcal{M}^* \to \mathcal{M}(A)^*$ is left adjoint[15] to the obvious forgetful pseudo-tensor functor $\mathcal{M}(A)^* \to \mathcal{M}^*$. To see this consider for $M \in \mathcal{M}(A)$ and $F_i, L_j \in \mathcal{M}$ the map $P_{I \sqcup J}^*(\{A \otimes F_i, L_j\}, M) \to P_{I \sqcup J}^*(\{F_i, L_j\}, M)$ sending an operation ψ to its composition with $1_A \otimes id_{F_i}\colon F_i \to A \otimes F_i$. This projection identifies the subspace of operations A-linear with respect to I variables with $P_{I \sqcup J}^*(\{F_i, L_j\}, M)$. The inverse map sends

[15] See 1.1.5.

$\varphi \in P^*_{I\sqcup J}(\{F_i, L_j\}, M)$ to the composition of $\varphi \otimes_I (\otimes id_A) \in P^*_{I\sqcup J}(\{A\otimes F_i, L_j\}, A^{\otimes I}\otimes M)$ with the product morphism $A^{\otimes I}\otimes M \to M$. If L_j are A-modules then our maps preserve operations A-linear with respect to J-variables, so $P^*_{A\ I\sqcup J}(\{A\otimes F_i, L_j\}, M)$ identifies with the subspace of $P^*_{I\sqcup J}(\{F_i, L_j\}, M)$ that consists of operations A-linear with respect to the J variables.

In the setting of (1.4.6.2) the pseudo-tensor functor $f^* : \mathcal{M}(B)^* \to \mathcal{M}(A)^*$ is left adjoint to the obvious forgetful pseudo-tensor functor $\mathcal{M}(A)^* \to \mathcal{M}(B)^*$.[16]

(iii) If a $*$ pairing $\langle\quad\rangle \in P^*_2(\{L', L\}, \mathbf{1})$ in $\mathcal{M}$ is non-degenerate (see 1.4.2) then such is the corresponding pairing $\langle\quad\rangle_A \in P^*_2(\{A\otimes L', A\otimes L\}, A)$ in $\mathcal{M}(A)$.

1.4.7 Differential $*$ operations. Many important $*$ operations between A-modules are not A-polylinear but belong to a larger class of *A-differential $*$ operations* we are going to define.

Let $\mathcal{I} \subset A\otimes A$ be the ideal of the diagonal, i.e., the kernel of the product map $A\otimes A \to A$. For an A-module L consider $A\otimes L$ as an $A\otimes A$-module and set $L^{(n)} := A\otimes L/\mathcal{I}^{n+1}(A\otimes L) = A^{(n)} \underset{A}{\otimes} L$. Now assume we have $M \in \mathcal{M}(A)$ and $L_i \in \mathcal{M}$. Pick some $i_0 \in I$ and identify $P^*_I(\{L_i\}, M)$ with the subspace of operations in $P^*_I(\{L_i, A\otimes L_{i_0}\}, M)$ A-linear with respect to the i_0th variable (see Remark (ii) in 1.4.6). Assume that L_{i_0} is also an A-module. The operations vanishing on $\mathcal{I}^{n+1}(A\otimes L_{i_0})$ form a subspace of $P^*_I(\{L_i\}, M)$; its elements are *differential operations of order $\leq n$ with respect to the i_0th variable.* If $n = 0$ these are the same as A-linear operations.

If all L_i's are A-modules then the $*$ operations which are differential operations of some order with respect to each variable are called *A-polydifferential $*$ operations of finite order.* The space of such operations is $\underset{n}{\cup} P^*_{A\,I}(\{L_i^{(n)}\}, M) \subset P^*_{A\,I}(\{A\otimes L_i\}, M) = P^*_I(\{L_i\}, M)$. They are stable with respect to composition, so A-polydifferential $*$ operations form another pseudo-tensor structure on $\mathcal{M}(A)$.

1.4.8 Action of Lie* algebras on commutative$^!$ algebras. If L is a Lie* algebra, then an action of L on a commutative$^!$ algebra A[17] is an L-module structure $\tau \in P^*_2(\{L, A\}, A)$

[16] To see this replace in the above argument $\mathcal{M}$ by $\mathcal{M}(B)$.

[17] Or an L-action by derivations.

on A (as on an object of $\mathcal{M}^*$) such that $\cdot_A: A \otimes A \to A$, $1_A: \mathbf{1} \to A$ are morphisms of L-modules.

Let τ be a fixed action of L on A. Then for $M \in \mathcal{M}(A)$ a *τ-action of L on M*[18] is an L-module structure on M (as on an object of $\mathcal{M}^*$) such that $\cdot_M: A \otimes M \to M$ is a morphism of L-modules. The A-modules equipped with τ-actions form an abelian category $\mathcal{M}(A, L, \tau)$. One may combine 1.4.4, 1.4.6 to show that

$$\mathcal{M}(A, L, \tau) \textit{ is an abelian augmented compound tensor } k\textit{-category}$$

in a natural way. We denote it $\mathcal{M}(A, L, \tau)^{*!}$.

Namely, for $M_i, N \in \mathcal{M}(A, L, \tau)$ set $P^*_{A,L,I}(\{M_i\}, N) := P^*_{LI}(\{M_i\}, N) \cap P^*_{AI}(\{M_i\}, N)$ (the intersection is taken in $P^*_I(\{M_i\}, N)$). This is the $*$ pseudo-tensor structure on $\mathcal{M}(A, L, \tau)$. Note that $\underset{AI}{\otimes} M_i$ is a quotient L-module of $\underset{I}{\otimes} M_i$, so we define the ! tensor product on $\mathcal{M}(A, L, \tau)$ as $\underset{A}{\otimes}$. The $\otimes^I_{S,T}$ operations all come from $\mathcal{M}(A)^{*!}$.

We may consider A as an object of $\mathcal{M}(A, L, \tau)$; this is the unit object.

The $*$-augmentation functor on $\mathcal{M}(A, L, \tau)^{*!}$ is $h^{\mathbf{L}}$ (see 1.4.4).

1.4.9 The Chevalley complex II. Let us assume that the $*$ dual $L' := \mathcal{H}om^*(L, \mathbf{1})$ of our Lie* algebra L exists and the canonical pairing $\langle \quad \rangle \in P^*_2(\{L', L\}, \mathbf{1})$ is non-degenerate (see 1.4.2). Below we give another interpretation of the inner Chevalley complexes from Remark (ii) in 1.4.5. As in 1.4.5, we deal with the general super setting (skipping the prefix "super").

Consider first the graded commutative$^!$ algebra $\mathrm{Sym}(L'[-1])$. It carries a canonical action of the graded Lie* algebra $L_\dagger$ from 1.4.5 (we forget about the differentials at the moment). Namely, $L \subset L_\dagger$ acts according to the coadjoint action, and its complement $L[1] \subset L_\dagger$ acts on the generators $L'[-1]$ by $-\langle \quad \rangle \in P^*_2(\{L[1], L'[-1]\}, \mathbf{1})$. Now the graded algebra $\mathrm{Sym}(L'[-1])$ admits a unique differential (odd of degree 1, see 1.1.16) such that the action of $L_\dagger$ are compatible with the differentials. Explicitly, on the generators $L'[-1]$ it is equal to the composition $L'[-2] \xrightarrow{\nu} L'[-1] \otimes L'[-1] \xrightarrow{-1/2\cdot} \mathrm{Sym}^2(L'[-1])$ where ν is the

[18] Or a τ-compatible action.

morphism dual to the Lie* bracket.[19] We denote this DG commutative$^!$ algebra equipped with the $L_\dagger$-action by $\mathcal{C}(L)$.

Let M be any L-module. Consider M as an $L_\dagger$-module[20] via the projection $L_\dagger \to L_\dagger/L[1] = L$. Then $L_\dagger$ acts on the graded $\mathrm{Sym}(L'[-1])$-module $M \otimes \mathrm{Sym}(L'[-1])$ via the tensor product of the above actions. As above, $M \otimes \mathrm{Sym}(L'[-1])$ admits a unique differential such that the actions of $L_\dagger$ and $\mathcal{C}(L)$ are compatible with the differentials. Explicitly, on the generators M the differential $M \to M \otimes L'$ is the morphism dual to the L-action. We denote this DG $\mathcal{C}(L)$-module equipped with $L_\dagger$-action by $\mathcal{C}(L, M)$. We leave it to the reader to check that it coincides with the inner Chevalley complex from Remark (ii) of 1.4.5.

If L acts on a commutative$^!$ algebra A by derivations, then $L_\dagger$ acts on $A \otimes \mathrm{Sym}(L'[-1])$ also by derivations, and the Chevalley differential is a derivation. Thus $\mathcal{C}(L, A)$ is a DG commutative$^!$ algebra equipped with a DG $L_\dagger$-action. If M is an A-module equipped with a compatible L-action (see 1.4.8) then $\mathcal{C}(L, M)$ is a DG $\mathcal{C}(L, A)$-module equipped with a compatible $L_\dagger$-action.

Remark. The morphism $\nu : L' \to L' \otimes L'$ is a Lie cobracket. The duality isomorphism $P_2^*(\{L, M\}, M) \xrightarrow{\sim} \mathrm{Hom}(M, L' \otimes M)$ identifies $*$ actions of L on M with $!$ coactions of L' on M. So the category of L-modules in $\mathcal{M}^*$ coincides with the category of L'-comodules in $\mathcal{M}^!$. If M is an L-module then $\mathcal{C}(L, M)$ is the usual Chevalley complex of the Lie coalgebra L' with coefficients in L'-comodule M in the tensor category $\mathcal{M}^!$.

1.4.10 Lie* algebroids. For $A \in \mathcal{C}omu(\mathcal{M}^!)$ a *Lie* A-algebroid*[21] is an A-module L, equipped with the following extra structures:

(a) a Lie* bracket $[\ \ ] \in P_2^*(\{L, L\}, L)$

(b) an action $\tau = \tau_L \in P_2^*(\{L, A\}, A)$ of L (considered as a Lie* algebra) on A

[19] So if $\langle\ \ \rangle^2 \in P_3^*(\{L' \otimes L', L, L\}, \mathbf{1})$ is the tensor product of 2 copies of $\langle\ \ \rangle$ then ν is uniquely determined by property $\langle\ \ \rangle^2(\nu, id_L, id_L) = \langle\ \ \rangle(id_{L'}, [\ \ ]) \in P_3^*(\{L', L, L\}, \mathbf{1})$.

[20] We forget about the differential on $L_\dagger$ at the moment.

[21] The definition of Lie algebroid in the usual (not compound) setting is recalled in 2.8.1 below; for more details see [M] or the first two sections of [BB].

that satisfy the following axioms:

(i) τ is A-linear with respect to the L-variable

(ii) The adjoint action ad_L is a τ-action of L (as a Lie* algebra) on L (as an A-module).

Notice that the structure $*$ operations (a), (b) are A-polydifferential (see 1.4.7).

Lie* A-algebroids form a category $\mathcal{LA}(A)$. Note that $\mathcal{L}ie(\mathcal{M}(A)^*)$ coincides with the full subcategory of $\mathcal{LA}(A)$ that consists of L with $\tau = 0$. The embedding $\mathcal{L}ie(\mathcal{M}(A)^*) \to \mathcal{LA}(A)$ admits a right adjoint functor $L \mapsto L^\circ := \operatorname{Ker}\tau$.

1.4.11. For $L \in \mathcal{LA}(A)$ an *L-module* M is an A-module M equipped with a τ-action $\tau_M \in P_2^*(\{L, M\}, M)$ of L (considered as a plain Lie* algebra) such that τ_M is A-linear with respect to L-variable. L-modules form a full subcategory[22] $\mathcal{M}(A, L) \subset \mathcal{M}(A, L, \tau)$ closed under subquotients and $\underset{A}{\otimes}$; clearly $A \in \mathcal{M}(A, L)$. Define the $*$ operations on $\mathcal{M}(A, L)$, as $P^*_{A,L}$, i.e., let us consider $\mathcal{M}(A, L)$ as a full augmented compound tensor subcategory $\mathcal{M}(A, L)^{*!}$ in $\mathcal{M}(A, L, \tau)^{*!}$. We see that

$$\mathcal{M}(A, L)^{*!} \textit{ is an abelian augmented compound tensor } k\textit{-category.}$$

Remark. The above notion of L-module is parallel to the one of *left* module for a usual Lie algebroid L_{us}. There seem to be no adequate compound analog of the notion of *right* L_{us}-module.

Let now E be any Lie* algebra, τ an action of E on A.

1.4.12 Lemma. (i) The induced A-module $A \otimes E$ has a unique structure of a Lie* A-algebroid such that the obvious morphism $1_A \otimes id_E : E \to A \otimes E$ is a morphism of Lie* algebras compatible with their actions on A. The functor $E \mapsto A \otimes E$ from the category of Lie* algebras acting on A to that of Lie* A-algebroids is left adjoint to the obvious "forgetting of A-action" functor.

(ii) The functor $\mathcal{M}(A, A \otimes E)^{*!} \to \mathcal{M}(A, E, \tau)^{*!}$ which assigns to an $A \otimes E$-module M the same M considered as an E-module is an equivalence of augmented compound tensor categories. □

[22] In the notation $\mathcal{M}(A, L, \tau)$ we consider L as a plain Lie* algebra acting on A.

1.4.13 The Chevalley complex III. As in 1.4.5, 1.4.9 we consider the super setting.

Let L be a Lie* A-algebroid. For $M \in \mathcal{M}(A,L)$ the A-polylinear operations form a subcomplex $C_A(L,M) \subset C(L,M)$ closed under the $h(L_\dagger)$-action. This is *the de Rham–Chevalley complex of L with coefficients in M*. Set $C_A(L) := C_A(L,A)$.

Assume now that the A-module L admits a dual A-module L' and the canonical $*$ pairing $\langle\ \ \rangle \in P^*_{A2}(\{L',L\},A)$ is non-degenerate (in $\mathcal{M}(A)^{*!}$, see 1.4.2). There is a unique action ad'_L of the Lie* algebra L on L' (called *the coadjoint action*) such that $\langle\ \ \rangle \in P^*_2(\{L',L\},A)$ is L-invariant (here L acts on L by the adjoint action, on A by τ).[23] It is easy to see[24] that ad'_L is a τ-action.

Consider the graded commutative$^!$ algebra $\mathrm{Sym}_A(L'[-1])$. The Lie* algebra L acts on it (via τ and ad'_L). This action extends to an action of the graded Lie* algebra $L_\dagger$ from 1.4.5.[25] Namely, $L[1] \subset L_\dagger$ kills A and acts on $L'[-1]$ by $\langle\ \ \rangle$.

We leave it to the reader to show that $\mathrm{Sym}_A(L'[-1])$ carries a unique differential d (odd of degree 1) such that the $L_\dagger$-action is compatible with the differentials. We denote the resulting DG commutative$^!$ algebra equipped with DG $L_\dagger$-action by $\mathcal{C}_A(L) = \mathcal{C}_A(L,A)$.

For $M \in \mathcal{M}(A,L)$ consider the graded $\mathrm{Sym}_A(L'[-1])$-module $M \underset{A}{\otimes} \mathrm{Sym}_A(L'[-1])$. It carries a natural action of the graded Lie* algebra $L_\dagger$ (namely, $L \subset L_\dagger$ acts according to the tensor product of the L-actions, and $L[1] \subset L_\dagger$ acts via the above action on $\mathrm{Sym}_A(L'[-1])$). Now $M \underset{A}{\otimes} \mathrm{Sym}_A(L'[-1])$ carries a unique differential such that the $L_\dagger$- and $\mathrm{Sym}_A(L'[-1])$-actions are compatible with the differentials. The resulting DG $\mathcal{C}_A(L)$-module equipped with DG $L_\dagger$-action is denoted by $\mathcal{C}_A(L,M)$. This is *the inner de Rham–Chevalley complex of L with coefficients in M*. We have $h(\mathcal{C}_A(L,M)) \xrightarrow{\sim} C_A(L,M)$ (cf. (1.4.5.1)).

It is easy to see that in situation of 1.4.12 the above construction amounts to that of 1.4.9: one has $\mathcal{C}_A(L) = \mathcal{C}(E,A)$ and $\mathcal{C}_A(L,M) = \mathcal{C}(E,M)$.

1.4.14 The tangent algebroid. For $M \in \mathcal{M}(A)$ we have the k-module of derivations

[23] Indeed, the operation $\langle\ \ \rangle(ad_L, id_{L'}) - \tau\langle\ \ \rangle \in P^*_3(\{L,L,L'\},A)$ is A-linear with respect to the second argument L, so, since $\langle\ \ \rangle$ is non-degenerate (use Remark (ii) from 1.4.6) there is a unique $ad'_L \in P^*_2(\{L,L'\},L')$ such that the above operation equals $\langle\ \ \rangle(id_L, ad'_L)$.

[24] Use universality property of $\langle\ \ \rangle$.

[25] We forget about the differential on $L_\dagger$ at the moment.

(with respect to ! tensor structure) $\mathrm{Der}(A,M) \subset \mathrm{Hom}_{\mathcal{M}}(A,M)$. The functor $\mathrm{Der}(A,\cdot)$ is representable: there is a universal derivation $d\colon A \to \Omega_A$.

More generally, for any finite collection of objects $\{N_i\}_{i\in I}$ of $\mathcal{M}$ we have a k-submodule $\mathrm{Der}(A,M)(\{N_i\})_I \subset P^*_{\tilde{I}}(\{N_i,A\},M)$ of operations that are derivations with respect to A, i.e., those φ the $\varphi(id_{N_i},\cdot_A) = \cdot_M(\varphi \otimes_1 id_A) + \cdot_M(\varphi \otimes_2 id_A) \in P^*_{\tilde{I}}(\{N_i, A\otimes A\},M)$. Here $\varphi \otimes_1 id_A$, $\varphi \otimes_2 id_A \in P^*_{\tilde{I}}(\{N_i, A\otimes A\},M)$ are the operations defined in 1.4.3; the labels 1,2 show which copy of A in $A\otimes A$ we insert in φ (the transposition of multiples symmetry of $A\otimes A$ interchanges them). Note that for any N_i-valued $*$ operations χ_i the composition $\varphi(\chi_i, id_A)$ is again a derivation with respect to A, and in case $I = \varnothing$ we have our old $\mathrm{Der}(A,M)$.

As follows from Remark (ii) in 1.4.6,[26] the map $P^*_{\tilde{I}}(\{N_i,\Omega_A\},M) \to P^*_{\tilde{I}}(\{N_i,A\},M)$, $\psi \mapsto \psi(id_{N_i},d)$, identifies the subspace of A-linear with respect to Ω_A operations with $\mathrm{Der}(A,M)(\{N_i\})_I$. If N_i are also A-modules, then this map identifies $P^*_{A\tilde{I}}(\{N_i,\Omega_A\},M)$ with the subspace of derivations A-linear with respect to N_i.

Assume that the object $\mathcal{D}er(A,M) := \mathcal{H}om_{\mathcal{M}(A)^*}(\Omega_A,M) \in \mathcal{M}(A)$ exists (see 1.3.12). By the above remark and Remark (ii) in 1.4.6 for any finite collection $\{G_i\}_{i\in I}$ one has a canonical identification $P^*_I\big(\{G_i\},\mathcal{D}er(A,M)\big) = \mathrm{Der}(A,M)(\{G_i\})_I \subset P^*_{\tilde{I}}(\{G_i,A\},M)$.

Let us show that $\Theta_A := \mathcal{D}er(A,A)$ (if it exists) is a Lie* A-algebroid in a natural way; this is *the tangent algebroid* of A. Namely, the action $\tau \in P^*_2(\{\Theta_A,A\},A)$ is the universal derivation (that corresponds to id_{Θ_A} under the above identification), and $[\ ] \in P^*_2(\{\Theta_A,\Theta_A\},\Theta_A)$ is uniquely defined by condition that τ is a Lie* action of Θ_A on A. Compatibilities 1.4.10 (i), (ii) are clear.

Remarks. (i) Θ_A is a universal Lie* A-algebroid: for any $L \in \mathcal{LA}(A)$ there is a unique morphism $L \to \Theta_A$.

(ii) Assume that the canonical $*$ pairing $\in P^*_{A2}(\{\Omega_A,\Theta_A\},A)$ is non-degenerate (see 1.4.2). Then for an A-module M a structure of Θ_A-module on M amounts to that of an integrable connection $M \to \Omega_A \underset{A}{\otimes} M$.

[26] And left exactness of P^*.

Let M be an A-module. One may look for a universal Lie* A-algebroid $\mathcal{E}(M)$ acting on M. If $\mathcal{E}(M)$ exists then $\mathcal{E}(M)^\circ \subset \mathcal{E}(M)$ is the Lie* A-algebra that corresponds to the associative* A-algebra $\mathcal{E}nd\, M$; we denote it $\mathfrak{gl}(M)$.

1.4.15 Coisson algebras. A *compound Poisson*, or simply *coisson*, *bracket* on $A \in \mathcal{C}omu(\mathcal{M}^!)$ is a Lie* bracket $\{\ \} \in P_2^*(\{A,A\},A)$ such that the adjoint action $\tau_{\{\ \}}$ is an action of A (considered as a Lie* algebra) on A (as on a commutative$^!$ algebra). We call such $(A, \{\ \})$ a *coisson algebra*.

By 1.4.12 (i) a coisson bracket $\{\ \}$ yields a Lie* A-algebroid structure on $A \otimes A$. One checks easily that the kernel of the projection $A \otimes A \to \Omega_A$, $a \otimes b \mapsto adb$, is an ideal in $A \otimes A$, so Ω_A inherits the Lie* A-algebroid structure. In other words, Ω_A carries a unique Lie* A-algebroid structure such that $d\colon A \to \Omega_A$ is a morphism of Lie* algebras and $\tau_{\Omega_A}(d, id_A) = \{\ \}$.

Conversely, a structure of Lie* A-algebroid on Ω_A yields a binary $*$ operation $\{\ \} := \tau_{\Omega_A}(d, id_A)$ on A. If $\{\ \}$ is skew-symmetric and $d\{\ \} = \{\ \}(d,d) \in P_2^*(\{A,A\},\ \Omega_A)$ then $\{\ \}$ is a coisson bracket. Therefore coisson brackets on A are the same as Lie* A-algebroid structures on Ω_A that satisfy the above conditions.

1.4.16. The category of coisson algebras $\mathcal{C}ois(\mathcal{M}^{*!})$ is a tensor category. Namely, for a finite family of coisson algebras $\{(A_i, \{\ \})\}_{i\in I}$, their tensor product is $(\underset{I}{\otimes} A_i, \{\ \})$ where $\{\ \} := \sum_I \{\ \}_i^\sim$ and $\{\ \}_{i_0}^\sim \in P_2^*(\{\underset{I}{\otimes} A_i, \underset{I}{\otimes} A_i\}, \underset{I}{\otimes} A_i)$ is the $\underset{I\smallsetminus\{i_0\}}{\otimes} A_i$-linear extension of $\{\ \}_{i_0}$ (see Remark (ii) in 1.4.6). The algebra $\mathbf{1}$ with zero coisson bracket is a unit object for this tensor structure.

Remark. There are obvious morphisms of coisson algebras $A_i \to \underset{I}{\otimes} A_i$ which mutually coisson commute in the obvious sense. This datum is universal: for any coisson algebra B the set $\mathrm{Hom}(\underset{I}{\otimes} A_i, B)$ is equal to the set of collections (φ_i) of mutually coisson commuting morphisms $\varphi_i : A_i \to B$.

1.4.17 Examples. (i) If L is a Lie* algebra then the symmetric$^!$ algebra $\mathrm{Sym}L$ carries a unique coisson bracket $\{\ \}$ such that $L = \mathrm{Sym}^1 L \hookrightarrow \mathrm{Sym}L$ is a morphism of Lie* algebras; this is the Kostant-Kirillov bracket. The functor $\mathcal{L}ie(\mathcal{M}^*) \to \mathcal{C}ois(\mathcal{M}^*), L \mapsto (\mathrm{Sym}L, \{\ \})$,

is left adjoint to the "forgetting of commutative$^!$ algebra structure" functor $\mathcal{C}ois(\mathcal{M}^{*!}) \to \mathcal{L}ie(\mathcal{M}^*)$.

(ii) More generally, if A is a commutative$^!$ algebra and L is a Lie* A-algebroid, then the symmetric$^!$ algebra $\mathrm{Sym}_A L$ of the A-module L carries a unique coisson bracket $\{\ \}$ such that $\{\mathrm{Sym}^p_A L, \mathrm{Sym}^q_A L\} \subset \mathrm{Sym}^{p+q-1}_A L$, and $\{\ \}$ coincides with $[\ ]$ on $\mathrm{Sym}^1 \times \mathrm{Sym}^1$ and with τ on $\mathrm{Sym}^1 \times \mathrm{Sym}^0$. For example, $Sym\Theta_A$ is the algebra of functions on the cotangent bundle to Spec$A$ and $\{\ \}$ is a canonical coisson structure on it.

(iii) The above constructions often come in twisted version. Let L be a Lie* algebra and $L^\flat$ a central extension of L by $\mathbf{1}$. The *twisted symmetric algebra* $\mathrm{Sym}^\flat L$ is the quotient of $\mathrm{Sym}(L^\flat)$ modulo the ideal generated by the image of the difference of embeddings $\mathbf{1} = \mathrm{Sym}^0(L^\flat) \hookrightarrow \mathrm{Sym}(L^\flat)$, $\mathbf{1} \subset L^\flat = \mathrm{Sym}^1(L^\flat) \hookrightarrow \mathrm{Sym}(L^\flat)$. This quotient algebra inherits the coisson bracket from $\mathrm{Sym}(L^\flat)$. Similarly, in the setting of (ii), if $L^\flat$ is a central extension of a Lie* algebroid L by A then the twisted symmetric algebra $\mathrm{Sym}^\flat_A L$ inherits a coisson bracket from $\mathrm{Sym}_A(L^\flat)$.

1.4.18 Hamiltonian reduction. The usual operations on Poisson algebras have obvious coisson versions. For example, if $(A, \{\ \})$ is a coisson algebra and $I \subset A$ is an ideal such that $\{I, I\} \subset I$, then I acts on A/I as a Lie* algebra. Set $B := (A/I)^I$ (:= the maximal I-invariant subobject of A/I; we tacitly assume that it exists). Then B is a ! subalgebra of A/I, and B inherits a coisson bracket $\{\ \}$ such that for $b_1, b_2 \in B$ and $\tilde{b}_i \in A$, $\tilde{b}_i \mathrm{mod} I = b_i$, one has $\{b_1, b_2\} = \{\tilde{b}_1, \tilde{b}_2\} \mathrm{mod} I$. In particular, for $(A, \{\ \}) \in \mathcal{C}ois(\mathcal{M}^{*!})$, $L \in \mathcal{L}ie(\mathcal{M}^*)$, and a morphism of Lie* algebras $\alpha\colon L \to A$, the commutative$^!$ algebra $B{:=}\ (A/\alpha(L)A)^L$ carries a canonical coisson bracket $\{\quad\}$ (here $\alpha(L)A$ is the ideal generated by $\alpha(L) \subset A$, so L acts on A via α and $\{\ \}$, $A/\alpha(L)A$ carries the quotient action, and $(\ )^L$ is the subobject of L-invariants). We call $(B, \{\ \})$ the *Hamiltonian reduction* of $(A, \{\ \})$.

1.4.19. Let $(A, \{\ \})$ be a coisson algebra. A *coisson* $(A, \{\ \})$*-module*, or simply $(A, \{\ \})$*-module*, is an A-module M (here A is considered as a commutative$^!$ algebra) together with a $\tau_{\{\ \}}$-action of A on M (here A is considered as a Lie* algebra) $\tau^{(A)}_M \in P^*_2(\{A, M\}, M)$ which is a derivation with respect to A-variable. The coisson modules form a category

$\mathcal{M}(A,\{\ \})$.

In fact $\mathcal{M}_{(A,\{\ \})}$ coincides with the category $\mathcal{M}(A,\Omega_A)$ where Ω_A is the Lie* A-algebroid from 1.4.15. The equivalence $\mathcal{M}(A,\Omega_A) \xrightarrow{\sim} \mathcal{M}(A,\{\ \})$ assigns to an Ω_A-module M same M with $\tau_M^{(A)}$ coming from the morphism of Lie* algebras $d\colon A \to \Omega_A$. Therefore $\mathcal{M}(A,\{\ \})$ carries a canonical augmented compound tensor structure; we denote it $\mathcal{M}(A,\{\ \})^{*!}$.

1.4.20 BRST (coisson version). This is a refined version of Hamiltonian reduction. Below we deal entirely with superobjects following conventions of 1.1.16; the prefix "super" is skipped. If M a graded superobject M we write $h(M)$ for the corresponding graded k-module, and $\hbar(M) := h(M)^{0even} = h(M^{0even})$. So $\hbar$ is the augmentation functor on the plain pseudo-tensor k-category $\mathcal{M}^{s\mathbb{Z}}$.[27] We assume that $k \supset \mathbb{Q}$.

Assume we have $L, L' \in \mathcal{M}^s$ and a $*$ pairing $\langle\ \ \rangle \in P_2^*(\{L',L\},\mathbf{1})$. Consider $L[1]\oplus L'[-1]$ as a commutative $\mathbb{Z}$-graded Lie* algebra. Let $(L[1]\oplus L'[-1])^\flat = L[1]\oplus L'[-1]\oplus \mathbf{1}$ be its central extension by $\mathbf{1}$ (placed in degree 0) with commutator $\in P_2^*(\{L'[-1],L[1]\},\mathbf{1}) = P_2^*(\{L',L\},\mathbf{1})$ equal to $\langle\ \ \rangle$. Set $\mathcal{C}l_c := \mathrm{Sym}^\flat(L'[-1]\oplus L[1])$ (see 1.4.17(iii)).[28] This coisson $\mathbb{Z}$-graded algebra is called the *Clifford coisson algebra*. One has:

(i) There are obvious embeddings $\mathrm{Sym}(L'[-1]), \mathrm{Sym}(L[1]) \hookrightarrow \mathcal{C}l_c$; the product morphism $\mathrm{Sym}(L'[-1]\oplus L[1]) = \mathrm{Sym}(L'[-1])\otimes \mathrm{Sym}(L[1]) \to \mathcal{C}l_c$ is an isomorphism of graded commutative$^!$ algebras.

(ii) The coisson bracket vanishes on both $L'[-1]$ and $L[1]$; the $\{\ \}$ pairing between $L'[-1]$ and $L[1]$ equals to $\langle\ \ \rangle$.

(iii) The subobject $L\otimes L' = L[1]\otimes L'[-1] \subset \mathcal{C}l_c^0$ is a Lie* subalgebra. The coisson bracket on it coincides with the bracket that comes from the associative* algebra structure as defined in 1.4.2. This subalgebra normalizes $L[1], L'[-1] \subset \mathcal{C}l_c^{\pm 1}$, and the corresponding adjoint action coincides with the action from 1.4.2.

Below we will need a simple lemma. For $A \in \mathcal{M}^s$ consider the coisson $\mathcal{C}l_c$-module $A\otimes \mathcal{C}l_c$. Let $F_c \subset h(A\otimes \mathcal{C}l_c)$ be the centralizer of $L[1] \subset \mathcal{C}l_c$. In other words, F_c is the

[27] With respect to even operations of degree 0, see 1.1.16.

[28] The lower index "c" is for "coisson" or "classical".

kernel of the morphism $\psi : h(A \otimes \mathcal{C}l_c) \to \mathrm{Hom}(L[1], A \otimes \mathcal{C}l_c)$ that comes from the adjoint action of $L[1]$ on $\mathcal{C}l_c$. It is clear that $F_c \supset h(A \otimes \mathrm{Sym}(L[1]))$.

1.4.21 Lemma. If $\langle \quad \rangle$ is non-degenerate (see 1.4.2) then $F_c = h(A \otimes \mathrm{Sym}(L[1]))$.

Proof. We have $h(A \otimes \mathcal{C}l_c^a) = \oplus h(A \otimes \mathrm{Sym}^n(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1]))$. Take any $\nu \in h(A \otimes \mathcal{C}l_c^a)$; let n be the largest number such that the nth component of ν is non-zero; denote it by $\bar{\nu} \in h(A \otimes \mathrm{Sym}^n(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1]))$. The morphism $\chi := \psi(\nu)$ sends $L[1]$ to the degree $a-1$ part of $A \otimes \mathrm{Sym}^{\leq n-1}(L'[-1]) \otimes \mathrm{Sym}(L[1])$; let $\bar{\chi}$ be its composition with the projection to $A \otimes \mathrm{Sym}^{n-1}(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1])$. Thus (see 1.4.2) $\bar{\chi} \in \mathrm{Hom}(L[1], A \otimes \mathrm{Sym}^{n-1}(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1])) = h(A \otimes (L'[-1] \otimes \mathrm{Sym}^{n-1}(L'[-1])) \otimes \mathrm{Sym}^{n-a}(L[1]))$. According to the Leibnitz formula the image of $\bar{\chi}$ by the product map $L'[-1] \otimes \mathrm{Sym}^{n-1}(L'[-1]) \to \mathrm{Sym}^n(L'[-1])$ equals $n\bar{\nu}$. If ν lies in F_c, i.e., $\chi = 0$, then $n\bar{\nu} = 0$, hence $n = 0$. □

1.4.22. ¿From now on L is a Lie* algebra. We assume that the $*$ dual $L' := \mathcal{H}om(L, \mathbf{1})$ exists and the canonical pairing $\langle \quad \rangle \in P_2^*(\{L', L\}, \mathbf{1})$ is non-degenerate (see 1.4.2). Then the adjoint action of L on L, hence on $L[1]$, yields a morphism of Lie* algebras $\beta : L \to L \otimes L' = L[1] \otimes L'[-1] \subset \mathcal{C}l_c^0$. We may consider β as an element of $\hbar(\mathcal{C}l_c^0 \otimes L')$. Denote by $\tilde{\beta}_c \in \hbar(\mathcal{C}l_c[1]) = h(\mathcal{C}l_c^{1\,odd})$ its image by the product map $\mathcal{C}l_c \otimes L'[-1] \to \mathcal{C}l_c$.

Let A be a coisson algebra, and $\alpha : L \to A$ a morphism of Lie* algebras. Consider the graded coisson algebra $A \otimes \mathcal{C}l_c^{\cdot}$. Let $L_\dagger$ be the DG Lie* algebra from 1.4.5. We have a morphism of graded Lie* algebras

$$\ell : L_\dagger \to A \otimes \mathcal{C}l_c \tag{1.4.22.1}$$

where $\ell^0 = \alpha + \beta : L \to A \otimes \mathcal{C}l^0$, ℓ^{-1} is the composition $L[1] \hookrightarrow \mathcal{C}l^{-1} \to A \otimes \mathcal{C}l^{-1}$.

Let us rewrite α as $\tilde{\alpha} \in h(A \otimes L')^{even} \subset \hbar(A \otimes \mathcal{C}l_c[1])$. Let $\tilde{\beta} \in \hbar(A \otimes \mathcal{C}l_c[1])$ be the image of $\tilde{\beta}_c$ by the morphism of algebras $\mathcal{C}l_c \to A \otimes \mathcal{C}l_c$. Set

$$\mathfrak{d}_c := \tilde{\alpha} + 1/2\,\tilde{\beta} \in \hbar(A \otimes \mathcal{C}l_c[1]). \tag{1.4.22.2}$$

This is *the (classical) BRST current.* Its adjoint action is an odd derivation d of degree 1 of the graded coisson algebra $A \otimes \mathcal{C}l_c$ called *the BRST differential.*

1.4.23 Proposition. (i) One has $\{\mathfrak{d}_c, \mathfrak{d}_c\} = 0$, so $d^2 = 0$.

We denote the DG coisson algebra $(A \otimes \mathcal{C}l_c, d)$ by $\mathcal{C}_{\mathrm{BRST}}\ (L, A)_c$.

(ii) ℓ commutes with the differentials, so we have a morphism of DG Lie* algebras

$$\ell : L_\dagger \to \mathcal{C}_{\mathrm{BRST}}\ (L, A)_c. \tag{1.4.23.1}$$

We refer to this as *the BRST property* of $\mathfrak{d}$ or d. It determines $\mathfrak{d}$ and d uniquely:[29]

(iii) $\mathfrak{d}_c$ is a unique element of $\hbar(A \otimes \mathcal{C}l_c[1])$ such that $[\mathfrak{d}_c, \ell^{-1}]$ equals ℓ^0, and d is a unique derivation of $A \otimes \mathcal{C}l$ of degree 1 such that $d\ell^{-1} = \ell^0$.

Proof. (i), (ii) are straightforward computations. (iii) follows from the next lemma. Let Der_c be the graded Lie algebra of derivations of the coisson algebra $A \otimes \mathcal{C}l_c$.

1.4.24 Lemma. The maps $h(A \otimes \mathcal{C}l_c^a) \to \mathrm{Der}_c^a \to \mathrm{Hom}(L^\Pi, A \otimes \mathcal{C}l_c^{a-1})$,[30] where the first arrow is the adjoint action, the second one is $\delta \mapsto \delta\ell^{-1}$, are injective for $a > 0$.

Proof. (a) Let δ be a coisson derivation of $A \otimes \mathcal{C}l_c$ of degree $a > 0$ such that $\delta\ell^{-1} = 0$, i.e., δ kills $L[1] \subset \mathcal{C}l_c$. Then it sends $L'[-1]$ and A to the centralizer of $L[1]$ in, respectively, $A \otimes \mathcal{C}l_c^{a+1}$ and $A \otimes \mathcal{C}l_c^a$, which is zero by 1.4.21. Thus $\delta = 0$.

(b) The kernel of the first arrow is contained in the centralizer of $L[1]$; by 1.4.21 it vanishes if $a > 0$. □

Considered as an object of an appropriate derived category $\mathcal{C}_{\mathrm{BRST}}(L, A)_c$ is called the *BRST reduction of* A with respect to α. Its cohomology form a graded coisson algebra $H^{\cdot}_{\mathrm{BRST}}(L, A)_c$. The reduction is *regular* if $H^{\neq 0}_{\mathrm{BRST}}(L, A)_c = 0$. Then the BRST reduction is a plain coisson algebra $H^0_{\mathrm{BRST}}(L, A)_c$.

1.4.25. Let us compare the BRST and Hamiltonian reductions (see 1.4.18). Set $\bar{A} := A/\alpha(L)A$. This is a commutative algebra; the L-action on A (via α and the coisson bracket) yields an L-action on $\bar{A}$. Let I be the DG ideal of $\mathcal{C}_{\mathrm{BRST}}(L, A)_c$ generated by the image of $\ell^{-1} : L[1] \to A \otimes \mathcal{C}l_c^{-1}$. The morphism of graded algebras $A \otimes \mathrm{Sym}(L'[-1]) \to A \otimes \mathcal{C}l_c/I$ is surjective; by 1.4.23 its kernel is the ideal generated

[29] This is a version of a theorem of Akman [A].

[30] For a $\mathcal{D}$-module M M^Π is a copy of M with the opposite pairity. The Hom is a super k-module.

by $\alpha(L)$. Thus $\mathcal{C}_{\mathrm{BRST}}(L,A)_c/I = \bar{A}\otimes \mathrm{Sym}(L'[-1])$. One checks immediately that the BRST differential yields on $\bar{A}\otimes \mathrm{Sym}(L'[-1])$ the usual Chevalley differential (see 1.4.9), so we have a canonical surjective morphism of DG commutative! algebras

$$\mathcal{C}_{\mathrm{BRST}}(L,A)_c \twoheadrightarrow \mathcal{C}(L,\bar{A}). \tag{1.4.25.1}$$

Passing to zero cohomology we get a canonical morphism of commutative algebras

$$H^0_{\mathrm{BRST}}(L,A)_c \to \bar{A}^L. \tag{1.4.25.2}$$

It is easy to see that it is compatible with the coisson brackets.

1.4.26. The compound tensor structure on $\mathcal{M}$ can be encoded in a single pseudo-tensor structure $\mathcal{M}^c$. Its construction below looks pretty much ad hoc at the moment; in 3.2 we will see that it presents a "classical limit" of chiral structure.

So let us bring the sundry polylinear operations of a compound tensor category under one roof. For an I-family $L_i\in\mathcal{M}$, $M\in\mathcal{M}$, and $S\in Q(I)$ (see 1.3.1) set

$$P^c_I(\{L_i\},M)_S := P^*_S(\{\underset{i\in I_s}{\otimes} L_i\},M)\otimes(\underset{s\in S}{\otimes}\mathcal{L}ie_{I_s}). \tag{1.4.26.1}$$

The k-module of c *operations* is

$$P^c_I(\{L_i\},M) := \underset{S\in Q(I)}{\oplus} P^c_I(\{L_i\},M)_S. \tag{1.4.26.2}$$

The c operations compose in a natural manner. Namely, assume we have $J\twoheadrightarrow I$, and a J-family of objects $K_j\in\mathcal{M}$. Take $\varphi\in P^c_I(\{L_i\},M)_S$, $\psi_i\in P^c_{J_i}(\{K_j\},M)_{T_i}$ (where $T_i\in Q(J_i)$). For $R\in Q(J)$ the R-component $\varphi(\psi_i)_R\in P^c_J(\{K_j\},M)_R$ vanishes unless $R\le T$,[31] I and R are transversal in $Q(T)$, and $\inf\{I,R\}=S$ (see 1.3.1). If these conditions are satisfied then $\varphi(\psi_i)_R=\varphi(\gamma_s)$ where for $s\in S$ one has[32] $\gamma_s := \otimes^{T_s}_{I_s,R_s}\psi_i \in P^*_{R_s}(\{\underset{j\in J_r}{\otimes}K_j\},\underset{i\in I_s}{\otimes}L_i)\otimes(\underset{t\in T_s}{\otimes}\mathcal{L}ie_{J_t})$ and the composition $\varphi(\gamma_s)$ is the tensor product of compositions of $*$ and Lie operations.[33] We leave it to the reader to check its associativity.

[31] Here $T:=\sqcup T_i\in Q(J)$.

[32] See 1.3.12 for notation.

[33] The composition of Lie operations is the tensor product of maps $\mathcal{L}ie_{I_s}\otimes(\underset{t\in T_r}{\otimes}\mathcal{L}ie_{J_t})\to\mathcal{L}ie_{J_r}$ labeled by $r\in R_s$. We use the fact that $T_r\xrightarrow{\sim} I_s$ (the transversality condition).

Therefore we defined an abelian[34] pseudo-tensor structure $\mathcal{M}^c$ on $\mathcal{M}$.

We define a natural $\mathbb{Z}$-grading on P^c as $P^c_I(\{L_i\},M)^n := \underset{S\in Q(I,|I|-n)}{\oplus} P^c_I(\{L_i\},M)_S$ where $Q(I,m)\subset Q(I)$ consists of quotients $I\twoheadrightarrow S$, $|S|=m$. Composition of c operations is compatible with the grading.

Notice that $P^*_I(\{L_i\},M)=P^c_I(\{L_i\},M)^0$ and $\mathrm{Hom}(\underset{I}{\otimes} L_i,M)\otimes \mathcal{L}ie_I = P^c_I(\{L_i\},M)^{|I|}$. The corresponding embedding and projection define the maps $P^!\otimes\mathcal{L}ie\to P^c\to P^*$ which are compatible with composition of operations, so we have the pseudo-tensor functors

$$\mathcal{M}^!\otimes\mathcal{L}ie\to\mathcal{M}^c\to\mathcal{M}^* \tag{1.4.26.3}$$

that extend the identity functor on $\mathcal{M}$.

Remark. It follows from 1.1.12 that the compound pseudo-tensor structure $\mathcal{M}^{*!}$ is uniquely determined by the pseudo-tensor structure $\mathcal{M}^c$ together with the $\mathbb{Z}$-grading on P^c.

1.4.27 Lemma. Lie algebras in $\mathcal{M}^c$ are the same as coisson algebras without unit in $\mathcal{M}^{*!}$.

Proof. A binary c operation $\mu\in P^c_2(M)$ is the same as a pair $(\cdot,\{\ \ \})$ where[35] $\cdot\in\mathrm{Hom}(M\otimes M,M)$ and $\{\ \ \}\in P^*_2(M)$. The skew-symmetry of μ amounts to symmetry of $\cdot$ and skew-symmetry of $\{\ \ \}$. The Jacobi identity for μ amounts to three identities (corresponding to the graded components):

– the Jacobi identity for $\cdot\otimes[\ \ ]$ which is the same as associativity of $\cdot$ (see 1.1.10),

– the Jacobi identity for $\{\ \ \}$,

– the identity which says that $\{\ \ \}$ acts by derivations of $\cdot$. □

[34] See 1.1.7.

[35] We use the identification $k\xrightarrow{\sim}\mathcal{L}ie_2$, $1\mapsto[\ \ ]$.

CHAPTER TWO: GEOMETRY OF $\mathcal{D}$-SCHEMES

Ломиться в двери пошлых аксиом,
Где лгут слова и красноречье храмлет?..

Б. Л. Пастернак, "Брюсову", 1926. †

In the following chapters the reader can understand "$\mathcal{D}$-module" as "super $\mathcal{D}$-module".[36]

§1 $\mathcal{D}$-MODULES: RECOLLECTIONS AND NOTATION

Below k denotes our base field of characteristic zero. All schemes, algebras, etc., live over k.

2.1.1. The details for 2.1.1 - 2.1.6 may be found in, say, [B]. For a scheme X we denote by $\mathcal{M}_{\mathcal{O}}(X)$ the category of $\mathcal{O}$-modules on X (:= quasicoherent sheaves of $\mathcal{O}_X$-modules). If X is smooth then $\mathcal{M}(X) = \mathcal{M}^r(X)$ (resp. $\mathcal{M}^\ell(X)$) denotes the category of right (resp. left) $\mathcal{D}$-modules on X (:= sheaves of $\mathcal{D}_X$-modules quasicoherent as $\mathcal{O}_X$-modules). We usually deal with right $\mathcal{D}$-modules, and call them simply $\mathcal{D}$-modules. One has the obvious forgetting of $\mathcal{D}$-action functors $o^r\colon \mathcal{M}(X) \to \mathcal{M}_{\mathcal{O}}(X)$, $o^\ell\colon \mathcal{M}^\ell(X) \to \mathcal{M}_{\mathcal{O}}(X)$.

If G, L are left $\mathcal{D}$-modules, then $G \otimes L = G \underset{\mathcal{O}_X}{\otimes} L$ is a left $\mathcal{D}$-module in a natural way. Therefore $\mathcal{M}^\ell(X)$ is a tensor category with a unit object $\mathcal{O}_X$, and o^ℓ is a tensor functor.

† Breaking into the doors of vulgar axioms
Where words are lying and the eloquence is limping?..
B.L. Pasternak, "To Bryusov", 1926.
We owe this motto to A. Polishchuk. Unfortunetely, the English rendition is poor:
(a) Russian "poshlyi" is not the same as "vulgar"; see Nabokov, "Strong Opinions", pp 100–101.
(b) the absence of an English equivalent for the used impossible conjugation form of the verb "limp" making it a derivative of the noun "temple".

[36] See 1.1.16.

If M is a right $\mathcal{D}$-module, then $L \otimes M = L \underset{\mathcal{O}_X}{\otimes} M$ is a right $\mathcal{D}$-module (a field $\tau \in \Theta_X \subset \mathcal{D}_X$ acts on $L \otimes M$ as $(\ell \otimes m)\tau = \ell \otimes m\tau - \tau\ell \otimes m$). The sheaf $\omega_X = \Omega_X^{\dim X}$ has a canonical right $\mathcal{D}$-module structure (namely, $\nu\tau = -\mathcal{L}ie_\tau(\nu)$ for $\nu \in \omega_X$). The functor $\mathcal{M}^\ell(X) \to \mathcal{M}(X)$, $L \mapsto L^r := L\omega_X = L \otimes \omega_X$, is an equivalence of categories. The inverse equivalence is $M \mapsto M^\ell := M \otimes \omega_X^{-1}$.

We call the identification of derived categories

$$D\mathcal{M}^\ell(X) \xrightarrow{\sim} D\mathcal{M}^r(X), \qquad L \mapsto L\omega_X[\dim X], \tag{2.1.1.1}$$

the canonical equivalence. So $D^b\mathcal{M}(X)$ carries two standard t-structures, $\mathcal{M}^r$- and $\mathcal{M}^\ell$-ones, that differ by shift by $\dim X$. We denote the composition $D^b\mathcal{M}(X) = D^b\mathcal{M}^\ell(X) \xrightarrow{o^\ell} D^b\mathcal{M}_{\mathcal{O}}(X)$ simply o^ℓ.

2.1.2. A morphism $f\colon X \to Y$ of smooth varieties yields canonical functors

$$D\mathcal{M}(X) \underset{f^!}{\overset{f_*}{\rightleftarrows}} D\mathcal{M}(Y)$$

that satisfy standard compatibilities (the base change property, adjunction property for proper f, etc.).

If f is quasi-finite then f_* is left exact, and if f is affine then f_* is right exact with respect to the $\mathcal{M}^r$ t-structure.

The diagram of functors

$$\begin{array}{ccc} D\mathcal{M}_{\mathcal{O}}(X) & \xleftarrow{Rf^!} & D\mathcal{M}_{\mathcal{O}}(Y) \\ \uparrow o^r & & \uparrow o^r \\ D\mathcal{M}(X) & \xleftarrow{f^!} & D^b\mathcal{M}(Y) \\ \downarrow o^\ell & & \downarrow o^\ell \\ D\mathcal{M}_{\mathcal{O}}(X) & \xleftarrow{Lf^*} & D\mathcal{M}_{\mathcal{O}}(Y) \end{array} \tag{2.1.2.1}$$

commutes; here $Rf^!$, Lf^* are standard pull-back functors for $\mathcal{O}$-modules (recall that for any $F \in D\mathcal{M}_{\mathcal{O}}(Y)$ one has $(Lf^*F) \otimes \omega_X[\dim X] = Rf^!(F \otimes \omega_Y[\dim Y])$). In particular,

$f^!$ is right exact with respect to the $\mathcal{M}^\ell$ t-structure. We often denote the corresponding functor $\mathcal{M}^\ell(Y) \to \mathcal{M}^\ell(X)$ by f^*.

If f is a smooth morphism then $f^!$ is exact with respect to the $\mathcal{M}^\ell$ t-structure. The left $\mathcal{D}$-modules satisfy smooth descent property, so the categories $\mathcal{M}^\ell(U)$, U is a smooth X-scheme, form a sheaf of abelian categories on the smooth topology X_{sm}, which we denote $\mathcal{M}^\ell(X_{sm})$. One defines the category $\mathcal{M}^\ell(Z)$ for any smooth algebraic stack Z in a standard way.

2.1.3. If $i\colon X \hookrightarrow Y$ is a closed embedding then i_* is exact with respect to the $\mathcal{M}^r$ t-structure, and its right adjoint $i^!$ is left exact. They define mutually inverse equivalences between $\mathcal{M}(X)$ and the full subcategory $\mathcal{M}(Y)_X \subset \mathcal{M}(Y)$ that consists of $\mathcal{D}_Y$-modules that vanish on $Y \setminus X$ (Kashiwara's lemma).

The above functors can be described expilicitely as follows. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the ideal of X. If $N \in \mathcal{M}(Y)$ then $i^!N$ coincides, as an $\mathcal{O}$-module, with the subsheaf of N that consists of sections killed by $\mathcal{I}$. For $M \in \mathcal{M}(X)$ one has $i_*M = M \underset{\mathcal{D}_X}{\otimes} (\mathcal{D}_Y/\mathcal{J}\mathcal{D}_Y)$. The identification $M \xrightarrow{\sim} i^!i_*M$ is $m \mapsto m \otimes 1$.

For $L \in \mathcal{M}^\ell(Y)$ and $M \in \mathcal{M}(X)$ there is a canonical isomorphism

$$i_*(i^*L \otimes M) \xrightarrow{\sim} L \otimes i_*M. \tag{2.1.3.1}$$

The corresponding isomorphism $i^*L \otimes M \xrightarrow{\sim} i^!(L \otimes i_*M)$ sends $l \otimes m$ to $l \otimes (m \otimes 1)$.

2.1.4. If X_i, $i \in I$, is a finite collection of smooth schemes then we have the exterior tensor product functors for both left and right $\mathcal{D}$-modules $\prod \mathcal{M}^r(X_i) \to \mathcal{M}^r(\prod X_i)$, $\prod \mathcal{M}^\ell(X_i) \to \mathcal{M}^\ell(\prod X_i)$, $(M_i) \mapsto \boxtimes M_i$. The canonical equivalence (2.1.1.1) identifies the corresponding derived functors.[37] The functors f_*, $f^!$ are compatible with $\boxtimes$: if $f_i\colon X_i \to Y_i$ are morphisms of smooth schemes then $(\prod f_i)_*(\boxtimes M_i) = \boxtimes(f_{i*}M_i)$, $(\prod f_i)^!(\boxtimes N_i) = \boxtimes(f_i^!N_i)$.

Assume that all X_i coincide with X; let $\Delta^{(I)}\colon X \to X^I$ be the diagonal embedding. For

[37] Notice that there is a canonical isomorphism $\omega_{\Pi X_i}[\dim \Pi X_i] \xrightarrow{\sim} \boxtimes \omega_{X_i}[\dim X_i]$; in particular, it does not depend on the ordering of I.

$L_i \in D\mathcal{M}^\ell(X)$ one has a canonical isomorphism

$$\overset{L}{\otimes} L_i = \Delta^{(I)!}(\boxtimes L_i) \tag{2.1.4.1}$$

2.1.5. If X is affine then projectivity of a $\mathcal{D}_X$-module is a local property for Zariski or étale topology of X (see theorem 2.3.7 below where this fact is proven in a more general situation). For arbitrary X we call a locally projective $\mathcal{D}_X$-module of finite rank a $\mathcal{D}_X$-*vector bundle* or a $\mathcal{D}$-*vector bundle on* X. Notice that a $\mathcal{D}_X$-vector bundle need not be a locally free $\mathcal{D}_X$-module as shows the following example.[38]

Example. Assume that X is an affine curve, and let $V \subset \mathcal{D}_X$ be a $\mathcal{D}$-submodule such that V equals $\mathcal{D}_X$ outside of a point $x \in X$. Then V is a projective $\mathcal{D}_X$-module (since the category of $\mathcal{D}$-modules has homological dimension 1). According to 2.1.13 below, such V's are the same as open k-*vector subspaces* of the formal completion O_x of the local ring $\mathcal{O}_x$.

If V is locally free then (shrinking X if necessary) we can assume that it is generated by a single section ϕ, i.e., $V = \phi\mathcal{D}_X \subset \mathcal{D}_X$. Since $\phi\mathcal{D}_X = \mathcal{D}_X$ on $X \smallsetminus \{x\}$, our ϕ is a function, i.e., $V = t^n\mathcal{D}_X$ for some $n \geq 0$ (where t is a parameter at x). Therefore locally free V's correspond to open *ideals* in O_x.

2.1.6. For $M \in \mathcal{M}(X)$ set $h(M) := M \underset{\mathcal{D}_X}{\otimes} \mathcal{O}_X = M/M\Theta_X$; this is a sheaf on X_{Zar}. For a local section $m \in M$ we denote by $\overline{m}$ its class in $h(M)$. Note that $h(M)$ is a presheaf on étale topology in the obvious way. In fact, it is a sheaf on $X_{\text{ét}}$. Denote by $DR(M)$ the de Rham complex of M, so $DR(M)^i := M \underset{\mathcal{O}_X}{\otimes} \bigwedge^{-i} \Theta_X$. One has $h(M) = H^0 DR(M)$; in fact, as an object of derived category of arbitrary sheaves on X_{Zar} or $X_{\text{ét}}$, $DR(M)$ equals to $M \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}_X$. Set $\Gamma_{DR}(X, M) := \Gamma(X, DR(M))$, $R\Gamma_{DR}(X, M) := R\Gamma(X, DR(M))$.

For a morphism $f\colon X \to Y$ of smooth schemes there is a canonical isomorphism $Rf_\cdot(DR(M)) \xrightarrow{\sim} DR(f_*M)$ in the derived category of sheaves on Y; here $f_\cdot$ is the sheaf-theoretical direct image, and $Rf_\cdot$ is its derived functor. For a more precise setting for this quasiisomorphism see 2.1.11. Two particular cases:

[38] We learned it from V.Ginzburg's talk about the work of Wilson on Calogero-Moser equation.

(a) The global de Rham cohomology coincide with the push-forward by projection $\pi: X \to (\text{point})$: one has $H^{\cdot}_{DR}(X, M) := H^{\cdot}(X, DR(M)) = H^{\cdot}\pi_* M$.

(b) If i: $X \hookrightarrow Y$ is a closed embedding then the above quasiisomorphism arises as a canonical embedding of complexes $i.DR(M) \hookrightarrow DRi_*(M)$ defined by the obvious embedding of $\mathcal{O}_Y$-modules $i.M \hookrightarrow i_* M$. Thus $i.h(M) = h(i_* M)$.

Notice also that in situation (b) for $N \in D\mathcal{M}(Y)$ the obvious morphism $DR(Ri^! N) \to Ri^! DR(N)$ is a quasiisomorphism.[39]

Lemma. (i) If M is a non-zero *coherent* $\mathcal{D}_X$-module then $h(M) \neq 0$.

(ii) If M is a coherent $\mathcal{D}_X$-module with no O_X-torsion then for every $\mathcal{D}_X$-module L the map $\mathrm{Hom}(L, M) \to \mathrm{Hom}(h(L), h(M))$ is injective.

Proof. (i) (a) Using induction by the dimension of the support of M we can assume, by (b) above, that M is non-zero at the generic point of X. Replacing X by its open subset we can assume that M is a free $\mathcal{O}_X$-module.[40]

(b) Let us show that for any (non necessary coherent) non-zero $\mathcal{D}_X$-module M which is free as an $\mathcal{O}_X$-module one has $h(M) \neq 0$. We use induction by $\dim X$. Replacing X by its open subset we can find a smooth hypersurface $i : Y \hookrightarrow X$ such that the determinant of the normal bundle to Y is trivial. Let $j : U \hookrightarrow X$ be its complement. Set $M_Y := i^!(j_* j^* M/M) = (i * M^\ell)^r$; this is a non-zero $\mathcal{D}$-module on Y which is free as an $\mathcal{O}_Y$-module by the condition on Y. Hence $h(M_Y) \neq 0$ by the induction hypothesis. Notice that $j.j^{\cdot} h(M) = h(j_* j^* M)$ and $h(i_* M_Y) = i.h(M_Y)$. Since h is right exact and $i_* M_Y$ is a quotient of $j_* j^* M$ we see that $h(M) \neq 0$.

(ii) (a) Since h is right exact it suffices to show that for every non-zero $L \subset M$ the map $h(L) \to h(M)$ is non-zero. The condition on M assures that we can replace X by any open subset, so we can assume that L, M, and M/L are free $\mathcal{O}_X$-modules.

(b) Let us show that for every (non-necessary coherent) non-zero $\mathcal{D}_X$-modules $L \subset M$ such that L, M, M/L are free $\mathcal{O}_X$-modules the map $h(L) \to h(M)$ is non-zero. We use

[39] Use the standard exact triangle $i_* Ri^! N \to N \to Rj_* j^* N \to i_* Ri^! N[1]$ where $j : Y \smallsetminus i(X) \hookrightarrow Y$.

[40] Indeed, choose a good filtration on M, then by EGA IV 6.9.2 $\mathrm{gr} M$ is $\mathcal{O}_X$-free over a non-empty open $V \subset X$.

induction by $\dim X$ and follow notation from (i)(b) above. It suffices to show that the map $h(j_*j^*L) \to h(j_*j^*M)$ is non-zero. It admits as a quotient the map $h(L_Y) \to h(M_Y)$ which is non-zero by induction by $\dim X$. □

2.1.7. The details for 2.1.7 – 2.1.13 may be found in [S]. Let X be a smooth scheme. For an $\mathcal{O}$-module B on X we have the induced right $\mathcal{D}$-module $B_{\mathcal{D}} := B \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \in \mathcal{M}(X)$. The de Rham complex $DR(B_{\mathcal{D}})$ is acyclic in degrees $\neq 0$, and the projection $DR(B_{\mathcal{D}})^0 = B \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to B$, $b \otimes \partial \mapsto b\partial(1)$, induces the isomorphism $h(B_{\mathcal{D}}) \xrightarrow{\sim} B$. In other words, there is a canonical quasiisomorphism

$$\nu_B \colon DR(B_{\mathcal{D}}) \to B. \tag{2.1.7.1}$$

If C is another $\mathcal{O}$-module then for any $\varphi \in \mathrm{Hom}_{\mathcal{M}(X)}(B_{\mathcal{D}}, C_{\mathcal{D}})$ the corresponding map $h(\varphi) \colon h(B_{\mathcal{D}}) \to h(C_{\mathcal{D}})$ is a differential operator $B \to C$. In fact, the map

$$h \colon \mathrm{Hom}(B_{\mathcal{D}}, C_{\mathcal{D}}) \to \mathrm{Diff}(B, C), \tag{2.1.7.2}$$

where Diff is the vector space of all differential operators, is an isomorphism. The inverse map assigns to a differential operator $\psi \colon B \to C$ the morphism $\psi_{\mathcal{D}} \colon B_{\mathcal{D}} \to C_{\mathcal{D}}$, $b \otimes \partial \mapsto \psi_b \partial$, where $\psi_b \in C \otimes \mathcal{D}_X = \mathrm{Diff}(\mathcal{O}_X, C)$ is the differential operator $\psi_b(f) = \psi(bf)$.

Let $\mathcal{D}iff(X)$ be the category whose objects are $\mathcal{O}$-modules on X and morphisms are differential operators. We see that the induction $B \mapsto B_{\mathcal{D}}$ defines a fully faithful functor $\mathcal{D}iff(X) \hookrightarrow \mathcal{M}(X)$.

If N is a left $\mathcal{D}$-module then for any $\mathcal{O}$-module B one has a canonical isomorphism of $\mathcal{D}$-modules $(N \underset{\mathcal{O}_X}{\otimes} B)_{\mathcal{D}} \xrightarrow{\sim} N \otimes B_{\mathcal{D}}$, $(n \otimes b) \otimes \partial \mapsto (n \otimes b)\partial$. Therefore $N, B \mapsto N \otimes B := N \underset{\mathcal{O}_X}{\otimes} B$ is an action of the tensor category $\mathcal{M}^\ell(X)$ on $\mathcal{D}iff(X)$; the induction functor $\cdot_{\mathcal{D}}$ commutes with $\mathcal{M}^\ell(X)$-actions.[41]

2.1.8 Examples. (i) Let E be an $\mathcal{O}$-module on X^n, $n \geqslant 2$, that vanishes outside the diagonal $\Delta \colon X \hookrightarrow X^n$. We may consider E as a sheaf on X equipped with n $\mathcal{O}_X$-module structures. The identity map $id_E \colon E \to E$ is a differential operator if the two copies of E

[41] Here $\mathcal{M}^\ell(X)$ acts on $\mathcal{M}(X)$ by the usual tensor product.

are considered with any two of the above $\mathcal{O}_X$-module structures. So E is a well-defined object of $\mathcal{D}iff(X)$. The corresponding $\mathcal{D}$-module coincides with $\Delta^!(E \underset{\mathcal{O}_{X^n}}{\otimes} \mathcal{D}_{X^n})$.

(ii) For a right $\mathcal{D}$-module M its de Rham complex $DR(M)$ is a complex in $\mathcal{D}iff(X)$. If L is a left $\mathcal{D}$-module then $DR(L \otimes M) = L \otimes DR(M)$.

(iii) Let $i: X \hookrightarrow Y$ be a closed embedding. For B as above consider the canonical embeddings of $\mathcal{O}_Y$-modules $i.B \hookrightarrow i.(B \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X) \hookrightarrow i_*(B_{\mathcal{D}})$. The composition induces a canonical morphism of $\mathcal{D}_Y$-modules $\gamma: (i.B)_{\mathcal{D}} \to i_*(B_{\mathcal{D}})$ which is an isomorphism.

2.1.9. For any $M \in \mathcal{M}(X)$ the de Rham complex $DR(M)$ is a complex in $\mathcal{D}iff(X)$ (the de Rham differential is a first order differential operator), so we have the corresponding complex $DR(M)_{\mathcal{D}}$ of $\mathcal{D}$-modules. It is acyclic in degrees $\neq 0$, and the projection $DR(M)^0_{\mathcal{D}} = M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to M$, $m \otimes \partial \mapsto m\partial$, defines an isomorphism $H^0 DR(M)_{\mathcal{D}} \xrightarrow{\sim} M$. In other words we have a canonical quasiisomorphism

$$\mu_M: DR(M)_{\mathcal{D}} \to M. \tag{2.1.9.1}$$

2.1.10. One may replace modules by complexes of those to get the functors between the DG categories of complexes

$$DR: C\mathcal{M}(X) \to C\mathcal{D}iff(X), \qquad \cdot_{\mathcal{D}}: C\mathcal{D}iff(X) \to C\mathcal{M}(X) \tag{2.1.10.1}$$

together with natural morphisms ν, μ as in 2.1.7.1, 2.1.9.1.

The category $\mathcal{D}iff(X)$ is not abelian. Following Saito, a morphism $\psi: B \to C$ of complexes in $\mathcal{D}iff(X)$ is called a quasiisomorphism if such is $\psi_{\mathcal{D}}$. The derived category $D^b\mathcal{D}iff(X)$ is defined by inverting quasiisomorphisms.

Remark. A morphism $\psi : B \to C$ in $C\mathcal{D}iff(X)$ is called a *naive quasiisomorphism* if it is a quasiisomorphism of complexes of plain sheaves of vector spaces. It is clear that ψ is a quasiisomorphism if and only if for any $\mathcal{O}$-flat left $\mathcal{D}$-module N the morphism $id_N \otimes \psi : N \otimes B \to N \otimes C$ is a naive quasiisomorphism. If both B, C are bounded above complexes of *coherent* $\mathcal{O}$-modules then every naive quasiisomorphism ψ is necessarily a

quasiisomorphism.[42] For general quasicoherent B, C this may be false (for example, take $B = DR(M)$ where M is a constant sheaf equal to the field of fractions of $\mathcal{D}_X$, $C = 0$).

Both μ and ν are quasiisomorphisms. They are compatible in the following sense: for any $B^{\cdot} \in K\mathcal{D}iff(X)$, $M^{\cdot} \in K\mathcal{M}(X)$ one has $DR(\mu_M) = \nu_{DR(M)}: DR(DR(M)_{\mathcal{D}}) \to DR(M)$, $(\nu_B)_{\mathcal{D}} = \mu_{B_{\mathcal{D}}}: DR(B_{\mathcal{D}})_{\mathcal{D}} \to B_{\mathcal{D}}$. Therefore DR and $\cdot_{\mathcal{D}}$ define mutually inverse equivalences between the derived categories:

$$D\mathcal{M}(X) \widetilde{\leftrightarrow} D\mathcal{D}iff(X). \tag{2.1.10.2}$$

For a morphism of schemes $f: X \to Y$ there is an obvious sheaf-theoretic push-forward functor $Rf.: D\mathcal{D}iff(X) \to D\mathcal{D}iff(Y)$. Equivalence (2.1.10.2) identifies it with f_*: there are canonical isomorphisms

$$DR(f_*M) = Rf.DR(M), \qquad f_*(B_{\mathcal{D}}) = (Rf.(B))_{\mathcal{D}}. \tag{2.1.10.3}$$

See particular cases in 2.1.8(iii), 2.1.6.

2.1.11 Remark. There seems to be no general way to find out whether a given D-module is induced or not. E.g. for an induced M its direct summand need not be induced (as shows the example from 2.1.5). A more handy category is that of *quasi-induced* $\mathcal{D}$-modules. One defines them as follows.

Let x be a (not-necessarily closed) point of X, $i_x : x \to X$ the embedding, and C an object of the derived category of $\mathcal{D}$-modules on X. Then $i_x^! C$ is defined to be the fiber of $i_U^! C$ over x where $U \subset X$ is a smooth locally closed subvariety containing x and i_U is the embedding $U \to X$. So $i_x^! C$ belongs to the derived category of $\mathcal{D}_x$-modules where $\mathcal{D}_x$ is the ring of differential operators on x. Let us say that C is *strongly non-negative* if for every $x \in X$ the Tor-dimension of $i_x^! C$ is non-negative. Equivalently, C is strongly non-negative if and only if for every $x \in X$ the Tor-dimension of the stalk C_x of C at x is $\leq \mathrm{codim} x$. It is easy to see that strong non-negativity implies the usual one (i.e., $H^i C = 0$

[42] Let M be the top non-zero cohomology of the cone of $\psi_{\mathcal{D}}$. Then M is a coherent $\mathcal{D}$-module such that $h(M) = 0$. We are done by the lemma from 2.1.6.

for $i < 0$) and that strong non-negativity is preserved under direct images with respect to arbitrary morphisms.

Remark. Assume that $k = \mathbb{C}$. A holonomic complex with regular singularities is strongly non-negative if and only if the complex of (usual, not perverse) sheaves corresponding to its the Verdier dual is non-positive.

We say that a $\mathcal{D}$-module on X is *quasi-induced* if it is strongly non-negative in the above sense. An induced $\mathcal{D}$-module is quasi-induced. The direct image of a quasi-induced $\mathcal{D}$-module under an affine morphism is quasi-induced.

2.1.12. Assume that $X \neq \emptyset$ is connected of dimension n.

We say that $M \in \mathcal{M}(X)$ is *constant* if M^ℓ is generated by global horizontal sections. The category of constant $\mathcal{D}$-modules is closed under subquotients. It identifies canonically with the category of vector spaces by means of the functors $M \mapsto \mathrm{Hom}(\mathcal{O}_X, M^\ell) = \Gamma^\nabla(X, M^\ell)$, $V \mapsto V \otimes \omega_X$.[43]

Assume that X is compact. Then every $M \in \mathcal{M}(X)$ admits the maximal constant quotient module $\pi : M \twoheadrightarrow M_{const}$.[44] The functor $M \mapsto M_{const}$ is right exact.

The adjunction formula for the projection from X to point (see 2.1.2) yields a canonical identification $M^\ell_{const} \xrightarrow{\sim} H^n_{DR}(X, M) \otimes \mathcal{O}_X$. For any point $s \in X$ the composition $H^n Ri^!_s M = M^\ell_s \xrightarrow{\pi_s} (M_{const})_s = H^n_{DR}(X, M)$ equals the n-th de Rham cohomology map for the canonical morphism $i_{s*}Ri^!_s M \to M$.

Let $S \subset X$ be any finite non-empty subset, $j_S : U \hookrightarrow X$ its complement. The standard exact triangle $\bigoplus\limits_S i_{s*}Ri^!_s M \to M \to Rj_{S*}j^*_S M \to \bigoplus\limits_S i_{s*}Ri^!_s M[1]$ shows that the kernel of the morphism $\pi_S = \oplus\pi_s : \bigoplus\limits_S M^\ell_s \to H^n_{DR}(X, M)$ is equal to the image of the "residue" map $\mathrm{Res}_S = \oplus\mathrm{Res}_s : H^{n-1}_{DR}(U, M) \to \bigoplus\limits_S M^\ell_s$.

2.1.13. Let $x \in X$ be a (closed) point; denote by i_x the embedding $\{x\} \hookrightarrow X$. For a $\mathcal{D}_X$-module M let $\Xi_x(M)$ be the set of all submodules $M_\xi \subset M$ such that M/M_ξ is supported

[43] To prove that these functors are mutually inverse notice that for a vector space V all sub-$\mathcal{D}$-modules of $V \otimes \mathcal{O}_X$ are of the form $W \otimes \mathcal{O}_X$, $W \subset V$; this follows from the irreducibility of the $\mathcal{D}$-module $\mathcal{O}_X$.

[44] A direct way to see the existence of M_{const}: reduce to the case where M^ℓ is coherent, then notice that $\dim \mathrm{Hom}_{\mathcal{D}}(M^\ell, \mathcal{O}_X) < \infty$.

at x. This is a subfilter in the ordered set of all submodules of M, so $\Xi_x(M)$ is a topology on M.

Consider the vector space $h(M)_x$. For $M_\xi \in \Xi_x(M)$ the sequence $0 \to h(M_\xi)_x \to h(M)_x \to h(M/M_\xi) \to 0$ is exact (since $H^{-1}_{DR}(M/M_\xi) = 0$). Therefore the map $N \mapsto h(N)_x \subset h(M)_x$ is a bijection between the set of $\mathcal{D}$-submodules $N \subset M$ containing M_ξ and that of vector subspaces of $h(M)_x$ containing $h(M_\xi)_x$ (since $\mathcal{D}$-submodules of M/M_ξ are the same as vector subspaces of $h(M/M_\xi)_x$). The subspaces $h(M_\xi)_x$, $M_\xi \in \Xi_x(M)$, form a topology on $h(M)_x$ which we call *the Ξ_x-topology.* We have seen that $M_\xi \mapsto h(M_\xi)_x$ is a bijection between $\Xi_x(M)$ and the set of open vector subspaces for the Ξ_x-topology.

Denote by $\hat{h}_x(M)$ the completion of $h(M)_x$ with respect to the Ξ_x-topology, i.e., the projective limit of the system of vector spaces $i^!_x(M/M_\xi) = h(M/M_\xi)_x$, $M_\xi \in \Xi_x(M)$. We consider $\hat{h}_x$ as a functor from $\mathcal{M}(X)$ to the category of complete separated topological vector spaces.[45] For any discrete vector space V one has[46] $\mathrm{Hom}_{cont}(\hat{h}_x(M), V) = \mathrm{Hom}(M, i_{x*}V)$, so the functor $\hat{h}_x$ is right exact.

Notice that if M is a coherent $\mathcal{D}$-module then $\hat{h}_x(M)$ is a profinite-dimensional vector space. Indeed, every M/M_ξ is a coherent $\mathcal{D}$-module supported at x, hence $i^!_x(M/M_\xi)$ is a finite-dimensional vector space.

We refer to any topology $\Xi^?_x(M)$ weaker than $\Xi_x(M)$ as a *topology at x on M.* As above, it yields a topology on $h(M)_x$. The corresponding completion $\hat{h}^?_x(M)$ equals $\varprojlim i^!_x(M/M_\xi)$, $M_\xi \in \Xi^?_x(M)$.

Examples. (i) Assume that M is an induced $\mathcal{D}$-module, $M = B_\mathcal{D}$ (see 2.1.7), so $h(M) = B$. Denote by $\Xi_x(B)$ the topology on B formed by all quasicoherent $\mathcal{O}$-submodules $P \subset B$ such that B/P is supported set-theoretically at x. Then $\Xi_x(M)$ is generated by submodules $P_\mathcal{D}$, $P \in \Xi_x(B)$.[47] Therefore $\hat{h}_x(M)$ is the completion of B with respect to the topology $\Xi_x(B)$. If B is $\mathcal{O}$-coherent then this is just the formal completion of B at x. We will

[45] In this work "topological vector space" always means vector space equipped with a *linear* topology (there is a base of neighbourhoods of 0 formed by linear subspaces).

[46] We would denote $\hat{h}_x(M)$ by $i^*_x M$ but, unfortunately, the notation was already used at the end of 2.1.2.

[47] For $M_\xi \in \Xi_x(M)$ set $P := M_\xi \cap B$. Then $P \in \Xi_x(B)$ and $M_\xi \supset P_\mathcal{D}$.

prove in 2.1.16 that for arbitrary B the $\Xi_x(B)$-completion of B equals $B \otimes O_x$ where $O_x := \varprojlim \mathcal{O}_x/\mathrm{m}_x^n$ is the formal completion of the local ring at x.

(ii) Assume that M is any coherent $\mathcal{D}$-module. To compute $\hat{h}_x(M)$ one can represent M as a cokernel of a morphism $\varphi : \mathcal{D}_X^m \to \mathcal{D}_X^n$. We have seen that $\hat{h}_x(\mathcal{D}_X) = O_x$, so $\hat{h}_x(M)$ is the cokernel of the corresponding matrix of differential operators[48] $\varphi : O_x^m \to O_x^n$.

2.1.14 Lemma. (i) $\hat{h}_x(M)$ is the completion of the vector space $M \underset{\mathcal{D}_X}{\otimes} O_x$ with respect to the topology formed by the images of the maps $M_\xi \underset{\mathcal{D}_X}{\otimes} O_x \to M \underset{\mathcal{D}_X}{\otimes} O_x$, $M_\xi \in \Xi_x(M)$.

(ii) If the restriction of M to $X \smallsetminus \{x\}$ is a coherent $\mathcal{D}$-module then the latter topology is complete and separated, i.e.,

$$M \underset{\mathcal{D}_X}{\otimes} O_x \xrightarrow{\sim} \hat{h}_x(M). \tag{2.1.14.1}$$

Proof. (i) If M is supported at x then $M \underset{\mathcal{D}_X}{\otimes} O_x$ is equal to the coinvariants of the right $\mathcal{D}_X$-module action of Θ_X on $M \underset{\mathcal{O}_X}{\otimes} O_x = M$, hence $M \underset{\mathcal{D}_X}{\otimes} O_x = h(M)_x$.

If M is arbitrary then for $M_\xi \in \Xi_x(M)$ we have $\mathrm{Coker}(M_\xi \underset{\mathcal{D}_X}{\otimes} O_x \to M \underset{\mathcal{D}_X}{\otimes} O_x) = (M/M_\xi) \underset{\mathcal{D}_X}{\otimes} O_x = h(M/M_\xi)_x$, which implies (i).

(ii) We want to show that, under the condition of (ii), the morphism of vector spaces $M \underset{\mathcal{D}_X}{\otimes} O_x \to \hat{h}_x(M)$ just defined is an isomorphism. Our M is an extension of a $\mathcal{D}$-module M' supported at x by a coherent $\mathcal{D}$-module M''. We have aready checked the statement for M' (see the proof of (i)) and M'' (see Example (ii) of 2.1.13). Since the sequences $0 \to M'' \underset{\mathcal{D}_X}{\otimes} O_x \to M \underset{\mathcal{D}_X}{\otimes} O_x \to M' \underset{\mathcal{D}_X}{\otimes} O_x \to 0$ and $0 \to \hat{h}_x(M'') \to \hat{h}_x(M) \to \hat{h}_x(M') \to 0$ are exact, we are done. □

Remarks. (i) If M is a $\mathcal{D}$-module as in 2.1.14(ii) then $\hat{h}_x(M)$ admits an open profinite-dimensional subspace.[49] Such topological vector spaces are discussed in 2.6.9 under the name of *Tate vector spaces*.

(ii) If M is an arbitrary $\mathcal{D}$-module then the map $\phi_M : M \underset{\mathcal{D}_X}{\otimes} O_x \to \hat{h}_x(M)$ need not be an isomorphism. Here is an example. Assume that $X = \mathrm{Spec}R$ is an affine curve, so

[48] Here we use the fact that a morphism of profinite-dimensional vector spaces has closed image.

[49] Namely, the image of $\hat{h}_x(M_\xi)$ where $M_\xi \in \Xi_x(M)$ is a coherent $\mathcal{D}_X$-module (as explained in 2.1.13, $\hat{h}_x(M_\xi)$ is profinite-dimensional).

$R = H^0_{DR}(X, \mathcal{D}_X) = \mathrm{Ext}^1_{\mathcal{M}(X)}(\omega_X, \mathcal{D}_X)$. Let M be the universal extension of $R \otimes \omega_X$ by $\mathcal{D}_X$. Then $M \underset{\mathcal{D}_X}{\otimes} O_x = O_x/R \neq 0$ and $\hat{h}_x(M) = 0$ is the completion of O_x/R with respect to the quotient topology, i.e., $\hat{h}_x(M) = 0$.

However ϕ_M is always an isomorphism if M is either an induced $\mathcal{D}$-module (see Example (i) of 2.1.13 and 2.1.16) or isomorphic to a direct sum of coherent $\mathcal{D}$-modules. In fact, the argument used in the proof of statement (ii) of the lemma shows that it suffices to assume that the restriction of M to $X \smallsetminus \{x\}$ satisfies either of these conditions.

(iii) Sometimes it is convemient to consider a finer structure than a plain topological vector space $\hat{h}_x(M)$. Namely, consider a directed set of all $\mathcal{D}$-coherent submodules $M_\alpha \subset M$ and set $\tilde{h}_x(M) := \varinjlim \hat{h}_x(M_\alpha)$. We consider $\tilde{h}_x(M)$ as an ind-object of the category of profinite-dimensional vector spaces. Recall that such an animal is the same as a left exact contravariant functor on the category of profinite-dimensional vector spaces with values in vector spaces. So $\tilde{h}_x(M)$ "represents" the functor $F \mapsto \varinjlim \mathrm{Hom}(F, \hat{h}_x(M_\alpha))$, F is a profinite-dimensional vector space. Notice that the dual object $\tilde{h}_x(M)^*$ – which is a pro-object of the category of vector spaces – equals $\varprojlim \mathrm{Hom}(M_\alpha, i_{x*}k)$.

2.1.15. We will also need a version of 2.1.13 in the case where x depends on parameters. So let Y be a scheme which we assume to be quasicompact and quasiseparated,[50] $x \in X(Y)$, $i_x : Y \hookrightarrow X \times Y$ the graph of x. Denote by $\mathcal{M}(X \times Y/Y)$ the category of right $\mathcal{D}_{X \times Y/Y} := \mathcal{D}_X \boxtimes \mathcal{O}_Y$-modules which are quasicoherent as $\mathcal{O}_{X \times Y}$-modules. Let $\mathcal{M}(X \times Y/Y)_x$ be the full subcategory of $\mathcal{D}_{X \times Y/Y}$-modules supported (set-theoretically) at the graph of x. A version of Kashiwara's lemma (see 2.1.3) says that the functor $i_x^!$ identifies $\mathcal{M}(X \times Y/Y)_x$ with the category of quasicoherent $\mathcal{O}_Y$-modules.

For $M \in \mathcal{M}(X \times Y/Y)$ we denote by $\Xi_x(M)$ the set of submodules $M_\xi \subset M$ such that $M/M_\xi \in \mathcal{M}(X \times Y/Y)_x$. This is a subfilter in the ordered set of all submodules of M. We have a $\Xi_x(M)$-projective system of quasicoherent $\mathcal{O}_Y$-modules $i_x^!(M/M_\xi)$ connected by surjective maps.

[50] The assumption is needed to assure that topological $\mathcal{O}_Y$-modules have local nature, i.e., proobjects in the category of quasicoherent $\mathcal{O}$-modules form a sheaf of categories on $Y_{\mathcal{Z}ar}$, and our constructions are compatible with localization of Y.

For $N \in \mathcal{M}(X \times Y/Y)$ set $h(N) := N \underset{\mathcal{D}_{X\times Y/Y}}{\otimes} \mathcal{O}_{X\times Y}$; this is a $p_Y^{\cdot}\mathcal{O}_Y$-module.[51] Since $i_x^!(M/M_\xi) = i_x^{\cdot}h(M/M_\xi)$ the projective limit of $i_x^!(M/M_\xi)$ equals to the completion of $i_x^{\cdot}h(M)$ with respect to the topology defined by the images of $i_x^{\cdot}h(M_\xi)$'s. We denote it by $\hat{h}_x(M)$.

If M is an induced module, $M = B \otimes \mathcal{D}_{X\times Y/Y}$ where B is an $\mathcal{O}_{X\times Y}$-module, then $h(M) = B$ and $\hat{h}_x(M)$ is the completion of B with respect to the topology formed by all quasicoherent $\mathcal{O}$-submodules $P \subset B$ which coincide with B outside of (the graph of) x. In particular, if B is $\mathcal{O}$-coherent then $\hat{h}_x(M)$ is the formal completion of B at x.

As in 2.1.14 for a coherent $\mathcal{D}_{X\times Y/Y}$-module M one identifies $\hat{h}_x(M)$ with the $\mathcal{D}_{X\times Y/Y}$-tensor product of M and the formal completion of $\mathcal{O}_{X\times Y}$ at x.

Remarks. (i) The above construction is compatible with base change by morphisms $f : Y' \to Y$: the obvious morphism of pro-$\mathcal{O}_{Y'}$-modules $\hat{h}_{xf}((f \times id_X)^*M) \to f^*\hat{h}_x(M)$ is an isomorphism.

(ii) If Y is a smooth variety and M is a right $\mathcal{D}_{X\times Y}$-module then differential operators on Y act on M in a continuous way with respect to Ξ_x-topology. Therefore $\hat{h}_x(M)$ is a sheaf of topological right $\mathcal{D}_Y$-modules. Similarly, if $\mathcal{D}_Y$ acts on M from the left then $\hat{h}_x(M)$ is a sheaf of topological left $\mathcal{D}_Y$-modules.

(iii) If F is a $\mathcal{D}$-module on Y and $\varphi : \hat{h}_x(M) \to F$ is a continuous $\mathcal{D}_Y$-morphism then it vanishes on the image of $\hat{h}_x(M_\xi)$ where $M_\xi \in \Xi_x(M)$ is a sub-$\mathcal{D}_{X\times Y}$-module of M. In particular, assume that $Y = X$, $x = id_X$, so $i_x = \Delta$, and $M = L \boxtimes \mathcal{O}_X$ for a $\mathcal{D}$-module L on X. Then every morphism φ as above is trivial (if $\dim X \neq 0$).

2.1.16. In this subsection we prove a result in commutative algebra we refered to in Example (i) of 2.1.13 and Remark (ii) in 2.1.14. It will not be used in the rest of this work.

Let A be a noetherian ring, $I \subset A$ an ideal such that A is complete in I-adic topology. Let C be the category of A-modules and $C_0 \subset C$ the full subcategory of A-modules M such that each element of M is annihilated by some power of I. The *I-topology* on an A-module M is the weakest topology such that every A-linear map from M to a module

[51] Here $p_Y : X \times Y \to Y$ is the projection.

from C_0 equipped with the discrete topology is continuous. The set B_M of all submodules $\tilde{M} \subset M$ such that $M/\tilde{M} \in C_0$ is a base of neighborhoods of 0 for the I-topology. If M is finitely generated then the I-topology coincides with the I-adic topology.

Theorem. (i) If N is a submodule of an A-module M then the I-topology on M induces the I-topology on N.

(iii) If A has finite Krull dimension then every A-module M is complete and separated for the I-topology.

Proof. To prove (i) we have to show that for every $\tilde{N} \in B_N$ there is an $\tilde{M} \in B_M$ such that $\tilde{M} \cap N = \tilde{N}$. Let P be the set of pairs $(L, \tilde{L})$ where L is a submodule of M and $\tilde{L} \in B_L$. Equip P with the following ordering: $(L, \tilde{L}) \leq (L', \tilde{L}')$ if $L \subset L'$ and $\tilde{L} = L \cap \tilde{L}'$. By Zorn's lemma there is a maximal $(L, \tilde{L}) \in P$ such that $(L, \tilde{L}) \geq (N, \tilde{N})$. It remains to show that $L = M$. Indeed, if $x \in M$, $x \notin L$ then one can construct $(L', \tilde{L}') > (L, \tilde{L})$ by putting $L' := L + Ax$, $\tilde{L}' := L' + U$ where $U \subset Ax$ is an open submodule of Ax such that $L \cap U = \tilde{L} \cap Ax$ (U exists because the I-topology on $L \cap Ax$ is induced by the I-topology on Ax).

Using (i) we reduce the proof of separatedness of M to the well known case where M is finitely generated (A need not have finite Krull dimension for this property).

Denote by $\hat{M}$ the completion of M with respect to the I-topology. Let $C^? \subset C$ be the full subcategory of A-modules M such that the map $M \to \hat{M}$ is bijective. We will show that $C^? = C$ if A has finite Krull dimension.

Lemma. $C^?$ has the following properties:

(i) Finitely generated modules belong to $C^?$.

(ii) Suppose that the sequence $0 \to M' \to M'' \to M \to 0$ is exact and $M \in C^?$. Then $M' \in C^? \Leftrightarrow M'' \in C^?$.

(iii) If $M_j \in C^?$, $j \in J$ then $\bigoplus_j M_j \in C^?$.

(iv) Suppose $M = \bigcup_i M_i$, $i = 1, 2, \ldots$, $M_i \subset M_{i+1}$. If $M_i \in C^?$ for all i then $M \in C^?$.

Proof of Lemma. (i) is well known.

(ii) is a corollary of the left exactness of the functor $M \mapsto \hat{M}$, which is proved as

follows. If the sequence $0 \to M_1 \to M_2 \to M_3$ is exact then $\mathrm{Ker}(\hat{M}_2 \to \hat{M}_3)$ is the closure of M_1 in $\hat{M}_2$, i.e., the completion of M_1 with respect to the topology induced from the I-topology on M_2. We have proved that the induced topology is the I-topology on M_1, so $\mathrm{Ker}(\hat{M}_2 \to \hat{M}_3) = \hat{M}_1$.

To prove (iii) it suffices to show that if $M = \bigoplus_j M_j$ then the image of the natural embedding $f : \hat{M} \to \prod_j \hat{M}_j$ equals $\bigoplus_j \hat{M}_j$. We only have to prove that $\mathrm{Im}f \subset \bigoplus_j \hat{M}_j$. Assume that $x = (x_j) \in \prod_j \hat{M}_j$ and $J' := \{j \in J | x_j \neq 0\}$ is infinite. Choose open submodules $U_j \subset M_j$ so that for every $j \in J'$ the image of x_j in M_j/U_j is nonzero. Then the image of x in $\prod_j M_j/U_j$ does not belong to $\oplus_j M_j/U_j$, so $x \notin \mathrm{Im}f$.

To prove (iv) it suffices to show that every $x \in \hat{M}$ belongs to the closure of $M_i \subset \hat{M}$ for some i (this closure equals $\hat{M}_i = M_i$). If this is not true then for every i there is an open submodule $U_i \subset M$ such that the image of x in $M/(M_i + U_i)$ is nonzero. We can assume that $U_1 \supset U_2 \supset \ldots$. The submodule $U := \bigcup_i (M_i \cap U_i) \subset M$ is open and $U + M_i \subset U_i + M_i$. So for every i the image of x in $M/(M_i + U)$ is nonzero, i.e., the image of x in M/U does not belong to the image of M_i in M/U. This is impossible because $M = \bigcup_i M_i$. □

We assume now that A has finite Krull dimension. It remains to show that if a full subcategory $C^? \subset C$ satisfies properties (i)–(iv) from the lemma then $C^? = C$. Let $C_i \subset C$ be the full subcategory consisting of modules M such that $M_\mathfrak{p} = 0$ for every prime ideal $\mathfrak{p} \subset A$ with $\dim A/\mathfrak{p} > i$. We will prove by induction that $C_i \subset C^?$. Suppose we know that $C_{i-1} \subset C^?$. Let $M \in C_i$. Denote by P_i the set of all prime $\mathfrak{p} \subset A$ such that $\dim A/\mathfrak{p} = i$. The map $M \to \prod_{\mathfrak{p} \in P_i} M_\mathfrak{p}$ factors through $\sum_{\mathfrak{p} \in P_i} M_\mathfrak{p}$, and the kernel and cokernel of the morphism $M \to \sum_{\mathfrak{p} \in P_i} M_\mathfrak{p}$ belong to C_{i-1}. By (ii),(iii), and the induction assumption, to prove that $M \in C^?$ it suffices to show that $M_\mathfrak{p} \in C^?$ for each $\mathfrak{p} \in P_i$. Let $\mathfrak{p} \in P_i$. Our $M_\mathfrak{p}$ is an $A_\mathfrak{p}$-module such that each element of $M_\mathfrak{p}$ is annihilated by a power of $\mathfrak{p}A_\mathfrak{p}$. By (ii) and (iv) we are reduced to proving that vector spaces over $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ belong to $C^?$. By (iii) it suffices to show that $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \in C^?$. The morphism $A/\mathfrak{p} \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is injective and its cokernel belongs to $C_{i-1} \subset C^?$. So $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \in C^?$ by (i) and (ii). □

Remark. If A is a 1-dimensional local domain then the proof of statement (ii) of the above theorem is very simple. In this case every A-module M has a free submodule N such that $M/N \in C_0$, so we can assume that M is free. Then apply statement (iii) of the lemma.

§2 The compound tensor structure

In this section we describe a canonical augmented compound tensor structure on the category $\mathcal{M}(X)$, X is a smooth scheme. We have to define the tensor structure $\otimes^!$, the pseudo-tensor structure P^*, the compound tensor product maps $\otimes^I_{S,T}$ (see 1.3.13), and the augmentation functor h.

2.2.1 The ! tensor structure. The category $\mathcal{M}^\ell$ is a tensor category (see 2.1.1). We use the standard equivalence $\mathcal{M}(X) \xrightarrow{\sim} \mathcal{M}^\ell(X)$, $M \mapsto M^\ell := M\omega_X^{-1}$, to define the tensor product $\otimes^!$ on $\mathcal{M}(X)$, so we have $M_1 \otimes^! M_2 = M_1^\ell \underset{\mathcal{O}_X}{\otimes} M_2$. The object ω_X is a unit object for $\otimes^!$.

2.2.2. Consider the derived tensor product $\overset{L}{\otimes}{}^!$ on $D^b\mathcal{M}(X)$.[52] According to (2.1.4.1) for $M_i \in D^b\mathcal{M}(X)$ one has

$$\overset{L}{\otimes}{}^!_I M_i = \Delta^{(I)!}(\boxtimes(M_i[\dim X]))[-\dim X] = \Delta^{(I)!}(\boxtimes M_i) \otimes \lambda_I^{\otimes \dim X}[(|I|-1)\dim X]$$

where $\lambda_I := \det k^I$ (the last equality comes since $k[\dim X]^{\otimes I} = \lambda_I^{\otimes \dim X}[|I|\dim X]$). Therefore for $M_i \in \mathcal{M}(X)$ there is a canonical isomorphism

$$\otimes^!_I M_i = H^{(|I|-1)\dim X}\Delta^{(I)!}(\boxtimes M_i) \otimes \lambda_I^{\otimes \dim X}. \tag{2.2.2.1}$$

Example. Assume that $M_i = \omega_X$. One has $\omega_X^{\otimes^! I} = \omega_X$ and $\omega_X^{\boxtimes I} = \omega_{X^I} \otimes \lambda_I^{\otimes \dim X}$, so (2.2.2.1) comes from the identification $\omega_X = H^{(|I|-1)\dim X}\Delta^{(I)!}(\omega_{X^I})$.

---

[52] Notice that $\overset{L}{\otimes}{}^!$ is identified with the tensor product on $D^b\mathcal{M}^\ell(X)$ by the above "standard" t-exact equivalence, and *not* by the canonical equivalence 2.1.1.1.

2.2.3 The $*$ pseudo-tensor structure. For an I-family L_i of $\mathcal{D}$-modules, $I \in \mathcal{S}$, and a $\mathcal{D}$-module M set

$$P_I^*(\{L_i\}, M) := \mathrm{Hom}(\boxtimes L_i, \Delta_*^{(I)} M). \tag{2.2.3.1}$$

The elements of P_I^* are called $*$ *I-operations*. They have étale local nature: we have the sheaf $\underline{P}_I^*(\{L_i\}, M)$, $U \mapsto P_I^*(\{L_i\big|_U\}, M\big|_U)$, on the étale topology of X. For a surjective map $\pi\colon J \twoheadrightarrow I$ the composition map

$$P_I^*(\{L_i\}, M) \otimes (\otimes_I P_{J_i}^*(\{K_j\}, L_i)) \longrightarrow P_J^*(\{K_j\}, M)$$

sends $\varphi \otimes (\otimes \psi_i)$ to $\varphi(\psi_i)$ defined as the composition

$$\boxtimes_J K_j \xrightarrow{\boxtimes \psi_i} \boxtimes_I \Delta_*^{(J_i)} L_i = \Delta_*^{(\pi)} (\boxtimes_I L_i) \xrightarrow{\Delta_*^{(\pi)}(\varphi)} \Delta_*^{(\pi)} \Delta_*^{(I)} M = \Delta_*^{(J)} M$$

where $\Delta^{(\pi)} = \prod\limits_I \Delta^{(J_i)}\colon X^I \hookrightarrow X^J$. The composition is associative, so P_I^* define on $\mathcal{M}(X)$ an abelian pseudo-tensor structure. We denote this pseudo-tensor category by $\mathcal{M}(X)^*$. The pseudo-tensor categories $\mathcal{M}(U)^*$, $U \in X_{\text{ét}}$, form a sheaf of pseudo-tensor categories on the étale topology of X.

Remark. One has $P_I^*(\{L_i\}, M) = \mathrm{Hom}(\otimes^* L_i, M)$ where $\otimes^* L_i := \Delta^{(I)*}(\boxtimes L_i)$ is the projective system of all the $\mathcal{D}_{X^I}$-module quotients of $\boxtimes L_i$ supported on the diagonal, considered as a projective system of $\mathcal{D}_X$-modules via the $\Delta^{(I)!}$-equivalence.

2.2.4 Examples. (i) Let us describe the $*$ pseudo-tensor structure on the subcategory of induced modules (see 2.1.7). For $\mathcal{O}$-modules $\{F_i\}_{i\in I}, G$ denote by $\mathrm{Diff}_I(\{F_i\}, G) \subset \mathrm{Hom}_k\Big(\underset{k}{\otimes} F_i, G\Big)$ the subspace of differential I-operations (those φ's for which the maps $F_i \to G$, $f_i \mapsto \varphi\Big(f_i \otimes \Big(\underset{j\neq i}{\otimes} f_j\Big)\Big)$, are differential operators for any i and any fixed local sections $f_j \in F_j$, $j \neq i$). Equivalently, $\mathrm{Diff}_I(\{F_i\}, G) = \mathrm{Diff}(\boxtimes F_i, \Delta_\cdot^{(I)} G)$.

The differential operations are stable with respect to composition so they define a pseudo-tensor structure on the category $\mathcal{D}iff(X)$; denote this pseudo-tensor category by $\mathcal{D}iff(X)^*$. One has $P_I^*(\{F_{i\mathcal{D}}\}, G_{\mathcal{D}}) = \mathrm{Hom}((\boxtimes F_i)_{\mathcal{D}}, (\Delta_\cdot G)_{\mathcal{D}}) = \mathrm{Diff}_I(\{F_i\}, G)$ (see 2.1.7, 2.1.8(iii)), i.e., $\cdot_{\mathcal{D}}$ extends to a fully faithful pseudo-tensor functor

$$\cdot_{\mathcal{D}}\colon \mathcal{D}iff(X)^* \hookrightarrow \mathcal{M}(X)^*. \tag{2.2.4.1}$$

(ii) Assume that $|I|\geq 2$ and one of the $\mathcal{D}$-modules L_i is lisse (e.g., equal to ω_X). Then $P_I^*(\{L_i\},M)=0$.

2.2.5. Let $i\colon X\hookrightarrow Y$ be an embedding of smooth schemes. The left exact functors H^0i_*, $H^0i^!$ between $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ act on P_I^* in the obvious way, i.e., they are pseudo-tensor functors. If Y is a closed subscheme then $H^0i^!$ is right adjoint to $i_*=H^0i_*$ as pseudo-tensor functors (which means that $P_I^*(\{i_*N_i\},M)=P_I^*(\{N_i\},H^0i^!M)$ for any $N_i\in\mathcal{M}(X)$, $M\in\mathcal{M}(Y)$), so i_* is a fully faithful embedding of pseudo-tensor categories.

2.2.6 The tensor product maps. We have to define the canonical maps (see 1.3.12)

$$\otimes^I_{S,T}\colon \otimes_S P^*_{I_s}(\{L_i\},M_s)\to P^*_T(\{\otimes^!_{I_t}L_i\},\otimes^!_S M_s)$$

According to the first formula in 2.2.2, one has canonical isomorphisms in $D^b\mathcal{M}(X^T)$ (here $\alpha:=k[-\dim X]$, so $\alpha^{-1}=k[\dim X]$):

$$\begin{aligned}\Delta^{(\pi_T)!}(\boxtimes_I L_i)=\boxtimes_T(\Delta^{(I_t)!}(\boxtimes_{I_t}L_i))&=\boxtimes_T((\overset{L}{\otimes}{}^!_{I_t}L_i)\otimes\alpha^{-1}\otimes\alpha^{\otimes I_t})\\&=(\boxtimes_T(\overset{L}{\otimes}{}^!_{I_t}L_i))\otimes(\alpha^{-1})^{\otimes T}\otimes\alpha^{\otimes I},\\ \Delta^{(\pi_T)!}(\boxtimes_S(\Delta_*^{(I_s)}M_s))=\Delta^{(\pi_T)!}\Delta_*^{(\pi_S)}(\boxtimes_S M_s)&=\Delta_*^{(T)}\Delta^{(S)!}(\boxtimes_S M_s)\\&=\Delta_*^{(T)}\left(\overset{L}{\otimes}{}^!_S M_s\right)\otimes\alpha^{-1}\otimes\alpha^{\otimes S}.\end{aligned}$$

In the second line we used the base change canonical isomorphism $\Delta^{(\pi_T)!}\Delta_*^{(\pi_S)}=\Delta_*^{(T)}\Delta^{(S)!}$ (recall that $X^S,X^T\subset X^I$ are transveral and $X^S\cap X^T=X$, see 1.3.2 (i)).

Consider the diagonal embeddings $\Delta^{(S)}\colon k\hookrightarrow k^S$, $\Delta^{(\pi_T)}\colon k^T\hookrightarrow k^I$. The embedding $\Delta^{(\pi_S)}\colon k^S\to k^I$ induces an isomorphism between the quotients $k^S/k\xrightarrow{\sim}k^I/k^T$. Since $\alpha^{-1}\otimes\alpha^{\otimes S}=\det((k^S/k)[-\dim X])$, $(\alpha^{-1})^{\otimes T}\otimes\alpha^{\otimes I}=\det((k^I/k)^T[-\dim X])$ we get the identification $\alpha^{-1}\otimes\alpha^{\otimes S}=(\alpha^{-1})^{\otimes T}\otimes\alpha^{\otimes I}$. Now $\otimes^I_{S,T}$ is the composition

$$\otimes_S P^*_{I_s}(\{L_i\},M_s)\to\mathrm{Hom}(\boxtimes_I L_i,\boxtimes_S\Delta_*^{(I_s)}M_s)\to\mathrm{Hom}(\boxtimes_T(\otimes^!_{I_t}L_i),\Delta_*^{(T)}(\otimes^!_S M_s))$$

where the first arrow is $\boxtimes_S$ and the second one is $H^{(|I|-|T|)\dim X}\Delta^{(\pi_T)!}$.

The commutativity of diagrams 1.3.12 (i), (ii) is clear. Therefore we have defined the compound tensor category structure on $\mathcal{M}(X)$. This compound tensor category $\mathcal{M}(X)^{*!}$ is abelian (see 1.3.14) with a strong unit object ω_X (see 1.3.16, 1.3.17, 2.2.4 (ii)).

The above constructions are compatible with étale localization therefore $\mathcal{M}(U)^{*!}$, $U \in X_{\text{ét}}$, form a sheaf of compound tensor categories $\mathcal{M}(X_{\text{ét}})^{*!}$ over $X_{\text{ét}}$.

2.2.7 The augmentation functor. Let us show that the de Rham functor $h = h_X$ (see 2.1.6) is a non-degenerate $*$-augmentation functor on $\mathcal{M}(X_{\text{ét}})^{*!}$ (see 1.2.5, 1.3.10). We have to define for a finite set I, $|I| \geq 2$, and $i_0 \in I$ a canonical morphism of sheaves

$$h_{I,i_0}\colon P_I^*(\{L_i\}, M) \otimes h(L_{i_0}) \to P^*_{I\smallsetminus\{i_0\}}(\{L_i\}, M) \tag{2.2.7.1}$$

Namely, for $\varphi \in P_I^*(\{L_i\}, M)$ and $\ell_i \in L_i$ one has

$$h_{I,i_0}(\varphi \otimes \bar{\ell}_{i_0})\Big(\underset{I\smallsetminus\{i_0\}}{\boxtimes} \ell_i\Big) = Tr_{i_0}\varphi\left(\underset{I}{\boxtimes} \ell_i\right). \tag{2.2.7.2}$$

Here $Tr_{i_0}\colon pr_{I,i_0}.\ \Delta_*^{(I)}M \to \Delta_*^{(I\smallsetminus\{i_0\})}M$ is the trace map for the projection $pr_{I,i_0}\colon X^I \to X^{I\smallsetminus\{i_0\}}$. It is easy to see that h_{I,i_0} is well-defined by (2.2.7.2). Compatibilities 1.2.5(i),(ii) and 1.3.10 are straightforward, so h makes $\mathcal{M}(X)^{*!}$ an augmented compound tensor category.

We leave it to the reader to check that h is non-degenerate, i.e., (see 1.2.5) that the morphisms $P_I^*(\{L_i\}, M) \to \mathrm{Hom}(h(L_{i_0}), P^*_{I\smallsetminus\{i_0\}}(\{L_i\}, M)$ coming from (2.2.7.1) are injective.

Let $Sh(X)^{\otimes}$ be the tensor category of sheaves of k vector spaces. According to 1.2.7, h extends to a pseudo-tensor functor

$$h : \mathcal{M}(X)^* \to Sh(X)^{\otimes}. \tag{2.2.7.3}$$

The maps $h_I\colon P_I^*(\{L_i\}, M) \to \mathrm{Hom}(\otimes_I h(L_i), h(M))$ are $h_I(\varphi)(\otimes\bar{\ell}_i) = \overline{\varphi(\boxtimes\ell_i)}$; here we consider $\overline{\varphi(\boxtimes\ell_i)}$ as a section of $\Delta^{(I)\cdot}h(\Delta_*^{(I)}M) = h(M)$ (see 2.1.6).

If $i\colon X \hookrightarrow Y$ is a closed embedding, then $i_*\colon \mathcal{M}(X)^* \hookrightarrow \mathcal{M}(Y)^*$ is compatible with the augmentation functors, i.e., it is an augmented pseudo-tensor functor (see 1.2.8).

Remark. The map $h_I: P_I^*(\{L_i\}, M) \to \mathrm{Hom}(\otimes_I h(L_i), h(M))$ can be not injective. However it is injective if M is either an induced $\mathcal{D}$-module (e.g. if M is supported at a single point) or a coherent $\mathcal{D}_X$-module. Indeed, in the induced case by left exactness of h we can assume that L_i's are also induced (replace L_i by $L_{i\mathcal{D}} \twoheadrightarrow L_i$ if necessary), and use then 2.2.4(i). In the coherent case use statement (ii) of Lemma from 2.1.6.

The rest of this section contains an auxiliary technical material; we suggest the reader to skip it returning when necessary.

2.2.8. Let us show that our augmented compound tensor structure comes from certain augmented compound strict $\mathcal{D}_X$-operad $\mathcal{B}^{*!}$ in the way explained in 1.3.18, 1.3.19.

Let us define $\mathcal{B}^{*!} = (\mathcal{B}^*, \mathcal{B}^!, <\ >^I_{S,T})$. For a finite set I let $\mathcal{B}^*_I$ be the tensor product of I copies of $\mathcal{D}_X$ considered as *left* $\mathcal{D}$-modules. This is a left $\mathcal{D}$-module on X, and the right $\mathcal{D}_X$-action on each copy of $\mathcal{D}_X$ defines the right $\mathcal{D}_X^{\otimes I}$-action on $\mathcal{B}^*_I$ (here $\mathcal{D}_X^{\otimes I}$ is the I^{th} tensor power of $\mathcal{D}_X$ considered as sheaf of k-algebras). So $\mathcal{B}^*_I$ is a $(\mathcal{D}_X - \mathcal{D}_X^{\otimes I})$-bimodule. Similarly, let $\mathcal{B}^!_I$ be the $\otimes^!$-product of I copies of $\mathcal{D}_X$ considered as *right* $\mathcal{D}_X$-modules; this is a $(\mathcal{D}_X^{\otimes I} - \mathcal{D}_X)$-bimodule.

Note that $\mathcal{B}^*_I$ is generated, as right $\mathcal{D}_X^{\otimes I}$-module, by the subsheaf $\mathcal{O}_X = \mathcal{O}_X 1^{\otimes I}$ on which $\mathcal{O}_X^{\otimes I} \subset \mathcal{D}_X^{\otimes I}$ acts via the product map $\mathcal{O}_X^{\otimes I} \to \mathcal{O}_X$. In fact $\mathcal{B}^*_I = \mathcal{O}_X \underset{\mathcal{O}_X^{\otimes I}}{\otimes} \mathcal{D}_X^{\otimes I}$. Similarly, $\mathcal{B}^!_I = \mathcal{D}_X^{\otimes I} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} (\omega_X^{(1-|I|)})$ as left $\mathcal{D}_X^{\otimes I}$-modules where $\omega_X^{(1-|I|)}$ is the $\mathcal{O}_X$-tensor power.

For a map $J \to I$ the composition morphisms for our augmented operads $\mathcal{B}^*_I \underset{\mathcal{D}_X^{\otimes I}}{\otimes} (\mathcal{B}^*_{J_i}) \to \mathcal{B}^*_J$, $(\otimes \mathcal{B}^!_{J_i}) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} \mathcal{B}^!_I \to \mathcal{B}^!_J$ are the isomorphisms of $(\mathcal{D}_X - \mathcal{D}_X^{\otimes J})$-, resp. $(\mathcal{D}_X^{\otimes J} - \mathcal{D}_X)$-bimodules, uniquely determined by the property that $f_I \otimes (\otimes f_{J_i}) \mapsto f_I \Pi f_{J_i} \in \mathcal{O}_X \subset \mathcal{B}^*_J$, $(\otimes \nu_{J_i}) \otimes \nu_I \mapsto (\Pi \nu_{J_i}) \nu_I \in \omega_X^{(1-|J|)} \subset \mathcal{B}^!_J$ for $f_I \in \mathcal{O}_X \subset \mathcal{B}^*_I$, $\nu_I \in \omega_X^{(1-|I|)} \subset \mathcal{B}^!_I$, etc.

It remains to define $<\ >^I_{S,T}: \left(\underset{S}{\otimes} \mathcal{B}^*_{I_s}\right) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} \left(\underset{T}{\otimes} \mathcal{B}^!_{I_t}\right) \to \mathcal{B}^!_S \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_T$. One has $\left(\underset{S}{\otimes} \mathcal{B}^*_{I_s}\right) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} \left(\underset{T}{\otimes} \mathcal{B}^!_{I_t}\right) = \mathcal{O}_X^{\otimes S} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} (\mathcal{D}_X \omega_X^{-1})^{\otimes I} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} \omega_X^{\otimes T}$ where the $\mathcal{O}_X^{\otimes I}$-module structure on $\mathcal{O}_X^{\otimes S}$, $\omega_X^{\otimes T}$ comes from the π_S and π_T product morphisms $\mathcal{O}_X^{\otimes I} \to \mathcal{O}_X^{\otimes S}$, $\mathcal{O}_X^{\otimes I} \to \mathcal{O}_X^{\otimes T}$. To see this $(\mathcal{D}_X^{\otimes S}, \mathcal{D}_X^{\otimes T})$-bimodule more clearly note that the left $\mathcal{D}_X^{\otimes I} \otimes \mathcal{D}_X^{\otimes I}$-module $(\mathcal{D}_X \omega_X^{-1})^{\otimes I}$

is the (sheaf-theoretic) restriction to the diagonal $X \subset X^I$ of the irreducible holonomic left $\mathcal{D}$-module $\mathcal{F}_I$ on $X^I \times X^I$ supported on $X^I \subset X^I \times X^I$ and generated by the $\mathcal{O}_{X^I}$-module $(\omega_X^{-1})^{\boxtimes I}$. Therefore, the left $\mathcal{D}_X^{\otimes S} \otimes \mathcal{D}_X^{\otimes T}$-module $\mathcal{O}_X^{\otimes S} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} (\mathcal{D}_X \omega_X^{-1})^{\otimes I} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} \mathcal{O}_X^{\otimes T}$ is the restriction to $X \subset X^S \times X^T$ of the pull-back $\mathcal{F}_{S,T}$ of $\mathcal{F}_I$ to $X^S \times X^T \subset X^I \times X^I$. This $\mathcal{F}_{S,T}$ is the irreducible holonomic left $\mathcal{D}$-module on $X^S \times X^T$ supported on $X^S \times X^T \cap X^I = X$ and generated by $\omega_X^{-|I|}$. Hence

$$\left(\underset{S}{\otimes} \mathcal{B}^*_{I_s}\right) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) = ((\Delta^{(S)}, \Delta^{(T)})^\cdot \mathcal{F}_{S,T}) \underset{\mathcal{O}_X^{\otimes T}}{\otimes} \omega_X^{\otimes T}$$

On the other hand, it is easy to see that the left $\mathcal{D}^{\otimes S} \otimes \mathcal{D}^{\otimes T}$-module $\mathcal{B}^!_S \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_T \underset{\mathcal{O}_X^{\otimes T}}{\otimes} (\omega_X^{-1})^{\otimes T}$ is the restriction to $X \subset X^S \times X^T$ of the same $\mathcal{F}_{S,T}$ (the generator is $\omega_X^{-|I|} = \omega_X^{1-|S|} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X \underset{\mathcal{O}_X^{\otimes T}}{\otimes} (\omega_X^{-1})^{\otimes T} \subset \mathcal{B}^!_S \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_T \underset{\mathcal{O}_X^{\otimes T}}{\otimes} (\omega_X^{-1})^{\otimes T}$). The resulting isomorphism $(\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) \xrightarrow{\sim} \mathcal{B}^!_S \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_T$ is our $< \quad >^I_{S,T}$.

The axioms of augmented compound operad are immediate.

2.2.9. Let us check that the augmented compound tensor structure defined on $\mathcal{M}(X)$ by $\mathcal{B}^{*!}$ (see 1.3.18, 1.3.19) coincides with that defined in 2.2.1 – 2.2.7. It is clear that $\otimes^!_I L_i = (\otimes L_i) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} \mathcal{B}^!_I$. The sheaf-theoretic restriction of the $\mathcal{D}_{X^I}$-module $\Delta^{(I)}_* M$ to the diagonal is a $\mathcal{D}_X^{\otimes I}$-module. There is a unique isomorphism of $\mathcal{D}_X^{\otimes I}$-modules

$$M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_I \xrightarrow{\sim} \Delta^{(I)\cdot} \Delta^{(I)}_* M \tag{2.2.9.1}$$

which induces the identity isomorphism between the subsheaves $M = M \otimes 1^{\otimes I}$ and $M = \Delta^{(I)!} \Delta^{(I)}_* M$. This yields the isomorphism

$$\mathrm{Hom}_{\mathcal{D}_X^{\otimes I}}(\otimes L_i, M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_I) \xrightarrow{\sim} P^*_I(\{L_i\}, M). \tag{2.2.9.2}$$

We leave it to the reader to identify the maps $< \quad >^I_{S,T}$.

2.2.10. We finish this section with several remarks about the $*$ pseudo-tensor structure. The operad $\mathcal{B}^*_I$ satisfies conditions (i), (ii) of 1.2.3. therefore, for any *coherent* $\mathcal{D}$-modules

L_i and an arbitrary $\mathcal{D}$-module M, the object $\mathcal{P}_I(\{L_i\}, M) \in \mathcal{M}(X)$ exists. Explicitly, one has

$$\mathcal{P}_I(\{L_i\}, M) = \underline{\mathrm{Hom}}_{\mathcal{D}_X^{\otimes I}}(\underset{k}{\otimes} L_i, M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}_{\widetilde{I}}^*). \tag{2.2.10.1}$$

Consider the morphism of $(\mathcal{O}_X, \mathcal{D}_X^{\otimes I})$-bimodules $\mathcal{B}_I^* \to \mathcal{B}_{\widetilde{I}}^*$, $b \mapsto 1. \otimes b$ (recall that $\widetilde{I} = \{\ \cdot\ \} \sqcup I$). The corresponding morphism of $(\mathcal{D}_X, \mathcal{D}_X^{\otimes I})$-bimodules $\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{B}_I^* \to \mathcal{B}_{\widetilde{I}}^*$ is isomorphism. It yields the identification $M \underset{\mathcal{O}_X}{\otimes} \mathcal{B}_I^* = M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}_{\widetilde{I}}^*$, hence also

$$\mathcal{P}_I(\{L_i\}, M) = \underline{\mathrm{Hom}}_{\mathcal{D}_X^{\otimes I}}(\underset{k}{\otimes} L_i, M \underset{\mathcal{O}_X}{\otimes} \mathcal{B}_I^*). \tag{2.2.10.2}$$

Note that the right $\mathcal{D}_X$-action on $M \underset{\mathcal{O}_X}{\otimes} \mathcal{B}_{\widetilde{I}}^*$ (used to define the $\mathcal{D}$-module structure on the $\underline{\mathrm{Hom}}$ sheaf) identifies with the right $\mathcal{D}$-module structure on $M \underset{\mathcal{O}_X}{\otimes} \mathcal{B}_I^*$ that comes from the right $\mathcal{D}_X$-module structure on M and the left one on $\mathcal{B}_I^*$ (see 2.1.1).

Remark. The functor $\mathcal{P}_I^*$ obviously commutes with inverse images for étale morphisms, i.e., it is defined on the sheaf of categories $\mathcal{M}(X_{\text{ét}})$. It also commutes with direct images for closed embeddings (as follows immediately from 2.2.5).

2.2.11. If we deal with induced $\mathcal{D}_X$-modules, i.e., $L_i = F_{i\mathcal{D}}$, $M = G_{\mathcal{D}}$, where F_i are coherent $\mathcal{O}$-modules, then the corresponding $\mathcal{D}$-module $\mathcal{P}_I^*$ can be described as follows. Consider the sheaf of differential I-operations $\mathrm{Diff}_I(\{F_i\}, G)_X$.[53] This is an object of $\mathcal{D}iff(X)$ by 2.1.8(i), so we have the corresponding $\mathcal{D}$-module $\mathrm{Diff}_I(\{F_i\}, G)_{\mathcal{D}} = \mathrm{Diff}_I(\{F_i\}, G)_{X\mathcal{D}}$. Now the canonical $\tilde{I}$-operation $\in \mathrm{Diff}_{\tilde{I}}(\{\mathrm{Diff}_I(\{F_i\}, G)_X, F_i\}, G) = P_{\tilde{I}}^*(\{\mathrm{Diff}_I(\{F_i\}, G)_{\mathcal{D}}, F_{i\mathcal{D}}\}, G_{\mathcal{D}})$ yields an isomorphism[54]

$$\mathrm{Diff}_{IX}(\{F_i\}, G)_{\mathcal{D}} \xrightarrow{\sim} \mathcal{P}_I^*(\{F_{i\mathcal{D}}\}, G_{\mathcal{D}}). \tag{2.2.11.1}$$

2.2.12. So for a $\mathcal{D}_X$-coherent L and any $M \in \mathcal{M}(X)$ we have the $\mathcal{H}om$ $\mathcal{D}_X$-module $\mathcal{H}om^*(L, M) = \underline{\mathrm{Hom}}_{\mathcal{D}_X}(L, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X) \in \mathcal{M}(X)$ equipped with a universal "evaluation"

[53] Which is the X-sheafified version of the vector space of differential I-operations from 2.2.4(i).

[54] To check that our arrow is an isomorphism one reduces (using left exactness of our functors) to the case $F_i = \mathcal{O}_X$ where the statement is obvious.

$*$ pairing $\varepsilon \in P_2^*\ (\{\mathcal{H}om^*(L,M),L\},M)$. If K is another coherent $\mathcal{D}$-module, then the composition $*$ pairing $c \in P_2^*(\{\mathcal{H}om^*(L,M),\mathcal{H}om^*(K,L)\},\mathcal{H}om^*(K,M))$ is defined; the composition is associative and compatible with evaluation (see 1.2.2).

Remark. ε is just the usual evaluation morphism $\underline{\mathrm{Hom}}\Big(L, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big) \underset{k}{\otimes} L \to M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$. To describe c explicitly, it is convenient to understand $\underline{\mathrm{Hom}}_{\mathcal{D}_X}\Big(L, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big)$ in the way different from what we used before: let us assume now that the $\mathcal{D}_X$-module structure on $M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ comes from the right $\mathcal{D}_X$-module structure on M and the left one on $\mathcal{D}_X$ while the action of $\mathcal{D}_X$ on $\underline{\mathrm{Hom}}_{\mathcal{D}_X}\Big(L, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big)$ comes from the right $\mathcal{D}_X$-action on $\mathcal{D}_X$. Since $M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X = M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_{\{1,2\}}$ has a canonical automorphism interchanging the two $\mathcal{D}$-module structures on $M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ we can still identify $\mathcal{H}om^*(L,M)$ with $\underline{\mathrm{Hom}}_{\mathcal{D}_X}\Big(L, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big)$. After this identification the $*$ pairing c becmes the usual composition

$$\underline{\mathrm{Hom}}\Big(L, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big) \underset{k}{\otimes} \underline{\mathrm{Hom}}\Big(K, L \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big) \to \underline{\mathrm{Hom}}\Big(K, \Big(M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big) \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big)$$
$$= \underline{\mathrm{Hom}}\Big(K, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big) \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X.$$

2.2.13. We see that for any coherent $M \in \mathcal{M}(X)$ the $\mathcal{D}$-module $\mathcal{E}nd^*(M) = \mathcal{H}om^*(M,M)$ is an associative algebra in $*$ sense acting on M. This action is universal: for any associative aglebra A in $\mathcal{M}(X)^*$ an A-module structure on M is the same as a morphism of associative algebras $A \to \mathcal{E}nd^*M$.

Set $M' := \mathcal{H}om^*(M, \omega_X)$; this is again a coherent module. We have a canonical pairing $\langle \quad \rangle \in P_2^*(\{M', M\}, \omega_X)$.

If M is a $\mathcal{D}_X$-vector bundle (see 2.1.5) then $\langle \quad \rangle$ is non-degenerate (see 1.4.2); in particular, $M \otimes M' \xrightarrow{\sim} \mathcal{E}nd^*M$.

2.2.14. According to 1.3.12, the duality functor $': \mathcal{M}(X)^\circ_{coh} \to \mathcal{M}(X)$ extends to the pseudo-tensor functor $': \mathcal{M}(X)^{\circ !}_{coh} \to \mathcal{M}(X)^*$. It induces an equivalence between the full pseudo-tensor subcategories of $\mathcal{D}_X$-vector bundles.

Example. Let F be a vector bundle on X, F^* the dual bundle, $F^\circ := F^* \otimes \omega_X$. Then $F_{\mathcal{D}}$ is a $\mathcal{D}_X$-vector bundle and there is a canonical identification $F'_{\mathcal{D}} := (F_{\mathcal{D}})' \xrightarrow{\sim} F^\circ_{\mathcal{D}}$

(defined by the image of the canonical pairing $\in \mathrm{Hom}(F\otimes F^\circ,\omega_X)\subset \mathrm{Diff}_2(\{F,F^\circ\},\omega_X)=P_2^*(\{F_{\mathcal{D}},F^\circ_{\mathcal{D}}\},\omega_{\mathcal{D}})$ by the canonical morphism $\omega_{\mathcal{D}}:=\omega_{X\mathcal{D}}\to\omega_X$). Denote by ${}_{\mathcal{D}}F$ the left $\mathcal{D}$-module $\mathcal{D}_X\underset{\mathcal{O}_X}{\otimes}F$; we see that $(F'_{\mathcal{D}})^\ell={}_{\mathcal{D}}F^*$.

If F_i, G are vector bundles on X then the duality isomorphism $':\mathrm{Hom}(G'_{\mathcal{D}},\otimes^! F'_{i\mathcal{D}})\xrightarrow{\sim}P_I^*(\{F_{i\mathcal{D}}\},G_{\mathcal{D}})$ is the composition of the following identifications (see 2.2.8 for notation) $\mathrm{Hom}({}_{\mathcal{D}}G^*,\otimes({}_{\mathcal{D}}F_i^*))=\mathrm{Hom}_{\mathcal{O}_X}(G^*,\mathcal{B}_I^*\underset{\mathcal{O}_X^{\otimes I}}{\otimes}(\otimes F_i^*))=\mathrm{Hom}_{\mathcal{O}_{XI}}(\boxtimes F_i,G\underset{\mathcal{O}_X}{\otimes}\mathcal{B}_I^*)=\mathrm{Hom}(\boxtimes F_{i\mathcal{D}},(\Delta_!^{(I)}G)_{\mathcal{D}})=\mathrm{Hom}(\boxtimes F_{i\mathcal{D}},\Delta_*^{(I)}G_{\mathcal{D}})$.

2.2.15. A remark about the augmentation functor. Take $I,J\in\mathbb{S}$, the families of objects $\{L_i\}$, $\{A_j\}$, M and consider the composition morphism for $I\hookrightarrow I\sqcup J$ in our augmented pseudo-tensor structure $\underline{P}^*_{I\sqcup J}(\{L_i,A_j\},M)\otimes\underset{J}{\otimes}h(A_j)\to\underline{P}^*_I(\{L_i\},M)$. We can rewrite it as a morphism

$$h_{IJ}\colon P^*_{I\sqcup J}(\{L_i,A_j\},M)\to\mathrm{Hom}\left(\underset{J}{\otimes}h(A_j),\underline{P}^*_I(\{L_i\},M)\right). \tag{2.2.15.1}$$

2.2.16 Lemma. The map $h_{I,J}$ is injective. Its image is the subspace of those $\varphi\colon\underset{J}{\otimes}h(A_j)\to P_I^*(\{L_i\},M)$ that the map $\widetilde{\varphi}\colon(\otimes L_i)\otimes(\otimes A_j)\to M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}_I^*$, $(\otimes\ell_i)\otimes(\otimes a_j)\mapsto\varphi(\otimes\bar{a}_j)(\otimes\ell_i)$, is a differential operator with respect to either of variables A_j (see 2.1.8(iii)).

In other words, φ satisfies the following condition (here $\mathcal{I}$ is the kernel of $\mathcal{O}_X^{\otimes I\sqcup J}\to\mathcal{O}_X$):

$(*)$ For any local section $s\in(\otimes L_i)\otimes(\otimes A_j)$ one has $\widetilde{\varphi}(\mathcal{I}^N s)=0$ for $N\gg 0$.

Proof. Let $Tr\colon M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}^*_{I\sqcup J}\to M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}^*_{I\sqcup J}\underset{\mathcal{D}_X^{\otimes J}}{\otimes}\mathcal{O}_X^{\otimes J}=M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}_I^*$ be the trace map along J-variables. The corresponding morphism of $\mathcal{D}_X^{\otimes I\sqcup J}$-modules $M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}^*_{I\sqcup J}\to\mathrm{Hom}_{\mathcal{D}_X^{\otimes I}}\left(\mathcal{D}_X^{\otimes I}\otimes\mathcal{O}_X^{\otimes J},M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}_I^*\right)$, which assigns to $b\in M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}^*_{I\sqcup J}$ the map $\alpha\mapsto Tr(b\alpha)$, is injective. Its image is the subspace of those $\psi\colon\mathcal{D}_X^{\otimes I}\otimes\mathcal{O}_X^{\otimes J}\to M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}_I^*$ that kill some $\mathcal{I}^N\subset\mathcal{O}_X^{\otimes I\sqcup J}\subset\mathcal{D}_X^{\otimes I}\otimes\mathcal{O}_X^{\otimes J}$. This morphism maps $P^*_{I\sqcup J}(\{L_i,A_j\},M)$ injectively to

$$\mathrm{Hom}_{\mathcal{D}_X^{\otimes I\sqcup J}}\left((\otimes L_i)\otimes(\otimes A_j),\mathrm{Hom}_{\mathcal{D}_X^{\otimes I}}\left(\mathcal{D}_X^{\otimes I}\otimes\mathcal{O}_X^{\otimes J},M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}_I^*\right)\right.$$
$$\left.=\mathrm{Hom}_{\mathcal{D}_X^{\otimes I}}\left((\otimes L_i)\otimes(\otimes h(A_j)),M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}_I^*\right)=\mathrm{Hom}(\otimes h(A_j),\underline{P}_I^*(\{L_i\},M))\right.$$

The image coincides with the subspace defined by condition $(*)$. Since this map coincides with $h_{I,J}$, we are done. $\square$

The following technical lemma will be of use. Let L_i, M be $\mathcal{D}$-modules on X, $\varphi \in P_I^*(\{L_i\}, M)$, $x \in X$ a closed point. Set $U_x := X \setminus \{x\}$. We use the terminology of 2.1.13.

2.2.17 Lemma. (i) The polylinear map $h(\varphi) : \otimes h(L_i) \to h(M)$ is Ξ_x-conitnuous with respect to each variable.

(ii) Assume that the restriction to U_x of each L_i to U_x is a countably generated $\mathcal{D}$-module. The $h(\varphi)$ is Ξ_x-continuous.

(iii) Assume that for certain $i_0 \in I$ every L_i but, possibly, L_{i_0} is a coherent $\mathcal{D}_X$-module. Then for each $M_\xi \in \Xi_x(M)$ there exists $L_{i_0\xi} \in \Xi_x(L_{i_0})$ such that

$$\varphi(L_{i_0\xi} \boxtimes (\underset{i \neq i_0}{\boxtimes} L_i)) \subset \Delta_*^{(I)} M_\xi.$$

Proof. (i) Pick $i_0 \in I$ and $\bar{l}_i \in h(L_i)_x$ for $i \neq i_0$. We have to show that the map $h(L_{i_0}) \to h(M)$, $\bar{l}_{i_0} \mapsto h(\varphi)(\bar{l}_{i_0} \otimes (\otimes \bar{l}_i))$, is Ξ_x-continuous. This is clear since this map comes from a morphism of $\mathcal{D}$-modules $L_{i_0} \to M$ that comes from φ and $\bar{l}_i$'s by the augmentation functor structure on h.

(ii) An induction by $|I|$ shows (use the augmentation functor structure on h) that to prove continuity of $h(\varphi)$ it suffices to find for every $M_\xi \in \Xi_x(M)$ some $L_{i\xi} \in \Xi_x(L_i)$ such that $h(\varphi)(\otimes L_{i\xi}) \subset M_\xi$. Shrinking X if necessary we can assume that X is affine and we have functions t_a, $a = 1, .., n$, on it which are local coordinates at x and have no other common zero. Let $l_{i1}, l_{i2}, ..$ be sections of L_i that generate the restriction of L_i to U_x as a $\mathcal{D}_{U_x}$-module. The quotient $\Delta_*^{(I)} M / \Delta_*^{(I)} M_\xi = \Delta_*(M/M_\xi)$ is a $\mathcal{D}_{X^I}$-module supported at point $\Delta^{(I)}(x)$, so every its section is killed by sufficiently high power of each t_a lifted from either copy of X. Therefore, using induction by $b = 1, 2, ..$, we can choose integers $n(a, b, i) \geq 0$ such that $h(\varphi)(\underset{I}{\boxtimes} t_a^{n(a,b_i,i)} l_{ib_i}) \in \Delta_*^{(I)} M_\xi \subset \Delta_*^{(I)} M$ for every $1 \leq b_i \leq b$. Our $L_{i\xi}$ is the $\mathcal{D}$-submodule of L_i generated by $\{t_a^{n(a,b,i)} l_{ib}\}$.

(iii) Choose for $i \neq i_0$ a finite set $\{l_{ij}\}$ of local sections of L_i at x that generates L_i. Now take for $L_{i_0\xi}$ the intersection of the preimages of $\Delta_*^{(I)} M_\xi$ by all maps $L_{i_0} \to \Delta_*^{(I)} M$, $l \mapsto \varphi(l \boxtimes (\boxtimes l_{ij}))$. $\square$

We see that in situation (ii) the morphism $h_I(\varphi)$: $\otimes h(L_i)_x \to h(M)_x$ extends by continuity to the map $\hat{h}_I(\varphi)_x$: $\widehat{\otimes}\,\hat{h}_x(L_i) \to \hat{h}_x(M)$. Here $\widehat{\otimes}\,\hat{h}_x(L_i) := \varprojlim \otimes h_x(L_i/L_{i\xi_i})$.

If we decide to play with better objects $\tilde{h}_x$ from Remark (iii) of 2.1.14 instead of plain topological vector spaces then the countability condition of (ii) becomes irrelevant.

§3 $\mathcal{D}_X$-SCHEMES

2.3.1. A *commutative $\mathcal{D}_X$-algebra* is a commutative unital $\mathcal{O}_X$-algebra equipped with an integrable connection along X. Unless it is stated explicitly otherwise our $\mathcal{D}_X$-algebras are assumed to be $\mathcal{O}_X$-quasicoherent. A *$\mathcal{D}_X$-scheme* is an X-scheme equipped with an integrable connection along X. Denote by $\mathcal{C}omu_{\mathcal{D}}(X)$, $\mathcal{S}ch_{\mathcal{D}}(X)$ the corresponding categories. We have a fully faithful functor Spec: $\mathcal{C}omu_{\mathcal{D}}(X)^{\circ} \hookrightarrow \mathcal{S}ch_{\mathcal{D}}(X)$; its essential image is the category of $\mathcal{D}_X$-schemes affine over X.

Remark. Let x_a be a coordinate system on X. Assume we have a system of non-linear partial differential equations $P_\alpha(u_i, \partial_a u_i, ...) = 0$ where P_α are polynomial functions with coefficients in functions on X. It yields a $\mathcal{D}_X$-algebra $R = R(\{P_\alpha\})$ defined as the quotient of the free $\mathcal{D}_X$-algebra $\mathrm{Sym}(\oplus \mathcal{D}_X \cdot u_i)$ modulo the relations P_α. A solution of our system with values in a (non necessary quasi-coherent) $\mathcal{D}_X$-algebra Φ is a collection of $\phi_i \in \Phi$ such that $P_\alpha(\phi_i, \partial_a \phi_i, ..) = 0$. This is the same as a morphism of $\mathcal{D}_X$-algebras $R \to \Phi$, $u_i \mapsto \phi_i$. Therefore $\mathcal{D}_X$-algebras provide a "coordinate free" languge for non-linear pde's.

Set $\mathcal{C}omu^!(X) := \mathcal{C}omu(\mathcal{M}^!(X))$ (we use the notation of 1.4.6). The equivalence of tensor categories $\mathcal{M}^!(X) \xrightarrow{\sim} \mathcal{M}^\ell(X)$, $A \mapsto A^\ell = A\omega_X^{-1}$, yields the equivalence

$$\mathcal{C}omu^!(X) \xrightarrow{\sim} \mathcal{C}omu_{\mathcal{D}}(X). \tag{2.3.1.1}$$

2.3.2. Denote by $\mathcal{S}ch(X)$, $\mathcal{C}omu(X)$ the category of X-schemes, resp. that of commutative $\mathcal{O}_X$-algebras (quasicoherent as $\mathcal{O}_X$-modules). We have the obvious forgetting of connection functors

$$o\colon \mathcal{C}omu_{\mathcal{D}}(X) \to \mathcal{C}omu(X), \quad \mathcal{S}ch_{\mathcal{D}}(X) \to \mathcal{S}ch(X). \tag{2.3.2.1}$$

They admit the left, resp. right, adjoint functors

$$\mathcal{C}omu(X) \to \mathcal{C}omu_{\mathcal{D}}(X), \quad \mathcal{S}ch(X) \to \mathcal{S}ch_{\mathcal{D}}(X). \tag{2.3.2.2}$$

We denote them both by $\mathcal{J}$. For $R \in \mathcal{C}omu(X)$ $\mathcal{J}R$ is generated by R. Explicitly, it is the $\mathcal{D}_X$-algebra quotient of the symmetric algebra $\mathrm{Sym}^{\cdot}\left(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} R\right)$ modulo the ideal generated by elements $\partial(1 \otimes r_1 \cdot 1 \otimes r_2 - 1 \otimes r_2 r_2) \in \mathrm{Sym}^2\left(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} R\right) + \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} R$ and $\partial(1 \otimes 1_R - 1) \in \mathcal{D}_X \otimes R + \mathcal{O}_X$ where $r_i \in R$, $\partial \in \mathcal{D}_X$, 1_R is the unit of R. This construction is compatible with localization. The functor $\mathcal{J}\colon \mathcal{S}ch(X) \to \mathcal{S}ch_{\mathcal{D}}(X)$ is $\operatorname{Spec} R \mapsto \operatorname{Spec} \mathcal{J}R$ for affine schemes; for arbitrary schemes use glueing. Note that for any $Z \in \mathcal{S}ch(X)$ the canonical projection $\mathcal{J}Z \to Z$ induces a bijection $(\mathcal{J}Z)_{\mathcal{D}}(X) \xrightarrow{\sim} Z(X)$ where $(\mathcal{J}Z)_{\mathcal{D}}(X)$ is the set of all horizontal sections $X \to Z$.

2.3.3. Here is a different description of $\mathcal{J}R$. Denote by $\mathcal{I}$ the kernel of the product map $\mathcal{O}_X^{\otimes 2} = \mathcal{O}_X \underset{k}{\otimes} \mathcal{O}_X \to \mathcal{O}_X$; for $n \geq 1$ set $\mathcal{O}_{X^{(n)}} := \mathcal{O}_X^{\otimes 2}/\mathcal{I}^n$. These $\mathcal{O}_{X^{(n)}}$ form a projective system of algebras; there are two obvious morphisms $\mathcal{O}_X \rightrightarrows \mathcal{O}_{X^{(n)}}$. Therefore any $\mathcal{O}_X$-algebra B yields two $\mathcal{O}_{X^{(n)}}$-algebras: $B \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{X^{(n)}}$ and $\mathcal{O}_{X^{(n)}} \underset{\mathcal{O}_X}{\otimes} B$.

2.3.4 Lemma. For any $R, B \in \mathcal{C}omu(X)$ one has

$$\mathrm{Hom}(\mathcal{J}R, B) = \varprojlim \mathrm{Hom}\left(\mathcal{O}_{X^{(n)}} \underset{\mathcal{O}_X}{\otimes} R, B \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{X^{(n)}}\right) \tag{2.3.4.1}$$

where the left and right Hom mean, respectively, the morphisms of $\mathcal{O}_X$- and $\mathcal{O}_{X^{(n)}}$-algebras.

Proof. Our statement is X-local, so we may assume that X is affine, $X = \operatorname{Spec} C$. Thus $\mathcal{O}_X$-algebras are the same as C-algebras. For a C-algebra B denote by J_B the kernel of the map $B \underset{k}{\otimes} C \to B$, $b \otimes c \mapsto bc$. Set $\mathcal{T}B := \varprojlim B \otimes C/J_B^n$. Let us consider $\mathcal{T}B$ as a C-algebra via the morphism $C \to \mathcal{T}B$, $c \mapsto 1 \otimes c$. The projection $\mathcal{T}B \to C$, $b \otimes c \mapsto bc$, is a morphism of C-algebras. Note that $\mathcal{T}B$ is a $\mathcal{D}_X$-algebra in the obvious way (the elments $b \otimes 1 \in \mathcal{T}B$ are horizontal). It is easy to see that the functor $\mathcal{C}omu(X) \to \mathcal{C}omu_{\mathcal{D}}(X)$, $B \mapsto \mathcal{T}B$, is *right* adjoint to the functor $o\colon \mathcal{C}omu_{\mathcal{D}}(X) \to \mathcal{C}omu(X)$. Therefore we have $\mathrm{Hom}(o\mathcal{J}R, B) = \mathrm{Hom}(\mathcal{J}R, \mathcal{T}B) = \mathrm{Hom}(R, o\mathcal{T}B)$. The latter set coincides with $\varprojlim \mathrm{Hom}$ in the statement of 2.3.4; we are done. □

So for any X-scheme Y the k-points of $\mathcal{J}Y$ are the same as pairs (x, γ), $x \in X$, γ is a section of Y on the formal neighborhood of x. We call $\mathcal{J}Y$ the *jet scheme of* Y.

Remark. The above constructions are compatible with étale localization, so we may replace schemes by algebraic spaces.

2.3.5. For $A^\ell \in \mathcal{C}omu_{\mathcal{D}}(X)$ we denote by $A^\ell[\mathcal{D}_X]$ the ring of differential operators with coefficients in A^ℓ. By definiton, this is a sheaf of associative algebras equipped with a morphism of algebras $A^\ell \to A^\ell[\mathcal{D}_X]$ and that of Lie algebras $\Theta_X \to A^\ell[\mathcal{D}_X]$ which satisfy the relations $\tau \cdot a - a \cdot \tau = \tau(a)$, $f \cdot \tau = f\tau$ for $a \in A^\ell$, $f \in \mathcal{O}_X \subset A^\ell$, $\tau \in \Theta_X$, and universal with respect to these data in the obvious sense. The embedding of Θ_X extends to the morphism of associative algebras $\mathcal{D}_X \to A^\ell[\mathcal{D}_X]$. One checks easily that the morphism of sheaves $A^\ell \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to A^\ell[\mathcal{D}_X]$, $a \otimes \partial \mapsto a \cdot \partial$ is an isomorphism.

For any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ we have the corresponding sheaf $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$ on $\mathcal{Y}_{\text{ét}}$.

2.3.6. An A^ℓ-module in the tensor category $\mathcal{M}^\ell(X)$ is the same as a left $A^\ell[\mathcal{D}_X]$-module quasicoherent as $\mathcal{O}_X$-module. Similarly, an A-module in $\mathcal{M}^!(X)$ is the same as right $A^\ell[\mathcal{D}_X]$-module quasicoherent as an $\mathcal{O}_X$-module. We denote by $\mathcal{M}^\ell(X, A^\ell)$, $\mathcal{M}(X, A^\ell) = \mathcal{M}^r(X, A^\ell)$ the categories of left, resp. right, $A^\ell[\mathcal{D}_X]$-modules. These are tensor categories (canonically identified) with unit object equal to A^ℓ, resp. A.

Every morphism of commutative $\mathcal{D}_X$-algebras $f : A^\ell \to B^\ell$ yields an adjoint pair of functors $f^* : \mathcal{M}(X, A^\ell) \to \mathcal{M}(X, B^\ell)$, $f_. : \mathcal{M}(X, B^\ell) \to \mathcal{M}(X, A^\ell)$ where $f^*M := B^\ell \underset{A^\ell}{\otimes} M$, $f_.N$ is N considered as an $A^\ell[\mathcal{D}_X]$-module. Notice that f^* is a tensor functor.

The faithfully flat descent works in any abelian tensor category, so $A^\ell[\mathcal{D}_X]$-modules are local objects with respect to flat topology.[55] So for any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ we have the corresponding categories $\mathcal{M}^\ell(\mathcal{Y})$, $\mathcal{M}^r(\mathcal{Y}) = \mathcal{M}(\mathcal{Y})$. There objects are sheaves of left, resp. right, $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules which are quasicoherent as $\mathcal{O}_Y$-modules. The categories $\mathcal{M}^\ell(\mathcal{U})$, $\mathcal{M}(\mathcal{U})$ for $\mathcal{U} \in \mathcal{Y}_{\text{ét}}$ form sheaves of categories $\mathcal{M}^\ell(\mathcal{Y}_{\text{ét}})$ on $\mathcal{Y}_{\text{ét}}$. One has $\mathcal{M}^\ell(\operatorname{Spec} A^\ell) = \mathcal{M}^\ell(X, A)$, $\mathcal{M}(\operatorname{Spec} A^\ell) = \mathcal{M}(X, A^\ell)$.

[55] f is faithfully flat as a morphism of commutative algebras in the tensor category $\mathcal{M}^\ell(X)$ if and only if it is faithfully flat as a morphism of plain $\mathcal{O}_X$-algebras.

Most of the facts mentioned in §2.1 render easily to the $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules setting. E.g., if G, L are left $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules and M is right one, then $G \underset{\mathcal{O}_\mathcal{Y}}{\otimes} L$ is left $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module in a natural way, and $L \underset{\mathcal{O}_\mathcal{Y}}{\otimes} M$ is right one. We have the standard equivalence $\mathcal{M}^\ell(\mathcal{Y}) \to \mathcal{M}^r(\mathcal{Y})$, $L \mapsto L\omega_X$. Any $N \in \mathcal{M}(\mathcal{Y})$ admits a canonical finite left resolution $(DR(N))_\mathcal{D}$ whose terms are induced $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules.

2.3.7 Theorem. Assume that $\mathcal{Y}$ is a $\mathcal{D}_X$-scheme which is affine (as a k-scheme). Then projectivity of an $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module is a local property for the Zariski or étale topology.

Proof. It occupies 2.3.8–2.3.10.

2.3.8. Let us first prove the theorem for a finitely generated $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module P. Suppose that P is projective over some étale covering of $\mathcal{Y}$. We need to show that P is projective. Take $N \in \mathcal{M}(\mathcal{Y})$. Consider the sheaf $\underline{\mathrm{Hom}}(P, N)$ on $\mathcal{Y}_{\text{ét}}$. It depends on N in exact way, so projectivity of P follows if we prove that $H^i(\mathcal{Y}_{\text{ét}}, \underline{\mathrm{Hom}}(P, N)) = 0$ for $i > 0$.

This is true if N is an induced module, $N = F \underset{\mathcal{O}_\mathcal{Y}}{\otimes} \mathcal{D}_X$. Indeed, in this situation $\underline{\mathrm{Hom}}(P, N)$ admits a structure of quasicoherent $\mathcal{O}_\mathcal{Y}$-module. Namely, the $\mathcal{O}_\mathcal{Y}$-module structure on $\underline{\mathrm{Hom}}(P, N)$ comes from an $\mathcal{O}_\mathcal{Y}$-action on N inherited from F. Quasicoherence is étale local property, so it suffices to check it over opens where P is projective. There $\underline{\mathrm{Hom}}(P, N)$ is isomorphic to a direct summand of a sum of several copies of N, which is obviously quasicoherent.

To prove acyclicity of $\underline{\mathrm{Hom}}(P, N)$ for arbitrary N use the fact that N admits a finite left resolution by induced modules (see 2.1.9, 2.3.6). We are done.

2.3.9. Without the finite generatedness assumption Theorem 2.3.7 is nontrivial even if X is a point. In this case it was conjectured by Grothendieck (see Remark 9.5.8 from [Gr]) and proved by Raynaud and Gruson (even for the fpqc topology); see [RG], p.82, 3.1.4. We will use the method of [RG]. It is based on the following fact: a flat module M over a ring R is projective if and only if M has the following properties:

(i) M is a direct sum of countably generated modules;

(ii) M is a Mittag-Leffler module

(the implication projectivity⇒(i) is due to Kaplansky [Ka]; the fact that (i) and (ii) imply projectivity is proved in [RG], p.74). A flat module M is said to be Mittag-Leffler if for some (or for every) representation of M as a direct limit of finitely generated projective modules P_i (i belongs to a directed ordered set I) the projective system (P_i^*) satisfies the Mittag-Leffler condition (i.e., for every $i \in I$ there exists $j \geq i$ such that $\mathrm{Im}(P_j^* \to P_i^*) = \mathrm{Im}(P_k^* \to P_i^*)$ for all $k \geq j$). Here $P_i^* := \mathrm{Hom}_R(P_i, R)$.

Now Theorem 2.3.7 is easily reduced to the following lemma.

2.3.10 Lemma. Assume that X is affine and $\mathcal{Y}$ is an affine $\mathcal{D}_X$-scheme. Then a flat $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module M is globally flat, i.e., $\Gamma(\mathcal{Y}, M)$ is flat over $\Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}[\mathcal{D}_X])$.

Proof. M has a finite left resolution consisting of globally flat $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules (indeed, M is $\mathcal{O}_\mathcal{Y}$-flat, so the canonical resolution $(DR(M))_\mathcal{D}$ has the required property). Therefore it suffices to show that if there is an exact sequence $0 \to F_1 \to F_0 \to M \to 0$ with F_0, F_1 globally flat and M flat then M is globally flat, i.e., for every left $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module L the morphism $\Gamma(\mathcal{Y}, F_1) \otimes_{\Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}[\mathcal{D}_X])} \Gamma(\mathcal{Y}, L) \to \Gamma(\mathcal{Y}, F_0) \otimes_{\Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}[\mathcal{D}_X])} \Gamma(\mathcal{Y}, L)$ is injective. The map $\Gamma(\mathcal{Y}, F_i) \otimes_{\Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}[\mathcal{D}_X])} \Gamma(\mathcal{Y}, L) \to \Gamma(\mathcal{Y}, F_i \otimes_{\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]} L)$ is an isomorphism (represent F_i as a direct limit of finitely generated free $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules). Finally, the morphism $F_1 \otimes_{\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]} L \to F_0 \otimes_{\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]} L$ is injective because M is flat. □

2.3.11. We see that for any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ there is a notion of $\mathcal{Y}$-locally projective $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module. Notice that for $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules on an affine $\mathcal{Y}$ the property of being finitely generated is obviously étale local. So we know what are $\mathcal{Y}$-locally finitely generated $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. We call a $\mathcal{Y}$-locally finitely generated and projective $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module a $\mathcal{D}_X$*-vector bundle on* $\mathcal{Y}$.

2.3.12. According to 1.4.6 (and §2.2) for a commutative $\mathcal{D}_X$-algebra A^ℓ the category $\mathcal{M}(X, A^\ell)$ carries a canonical augmented compound tensor structure. As in 1.4.6 for $L_i, M \in \mathcal{M}(X, A^\ell)$ we denote the corresponding space of A^ℓ-polylinear $*$ operations by $P^*_{A\,I}(\{L_i\}, M)$.

In fact, this structure comes from an augmented compound strict $A^\ell[\mathcal{D}_X]$-operad $\mathcal{B}^{*!}_A$. We leave the explicit construction of $\mathcal{B}^{*!}_A$ to the reader (repeat 2.2.8, replacing everywhere

$\mathcal{D}_X$ by $A^\ell[\mathcal{D}_X]$). The discussion 2.2.10 – 2.2.14 remains literally valid for $A^\ell[\mathcal{D}_X]$-modules if we replace the words "coherent $\mathcal{D}_X$-module" by "finitely presented $A^\ell[\mathcal{D}_X]$-module". The lemma 2.2.16 also remains valid.

According to Remark (i) of 1.4.6 for every morphism of commutative $\mathcal{D}_X$-algebras $f : B^\ell \to A^\ell$ the corresponding tensor functor f^* from 2.3.6 extends to a compound tensor functor (base change)

$$f^* : \mathcal{M}(X, A^\ell)^{*!} \to \mathcal{M}(X, B^\ell)^{*!}. \tag{2.3.12.1}$$

We also have a forgetful paseudo-tensor functor

$$f_* : \mathcal{M}(X, B^\ell)^* \to \mathcal{M}(X, A^\ell)^* \tag{2.3.12.2}$$

right adjoint to f^* considered as a $*$ pseudo-tensor functor (see Remark (ii) of 1.4.6).

Lemma. $\mathcal{M}(X, A^\ell)^{*!}$ has local nature with respect to flat topology of SpecA^ℓ.

Proof. We already discussed the flat descent of $A^\ell[\mathcal{D}_X]$-modules in 2.3.6, so it remains to check that the A-polylinear $*$ operations satisfy the flat descent property. For $L_i, M \in \mathcal{M}(X, A^\ell)$ the above adjunction identifies the descent data with operations $\psi \in P^*_{A\,I}(\{L_i\}, B^\ell \underset{A^\ell}{\otimes} M)$ whose composition with the two arrows $B^\ell \underset{A^\ell}{\otimes} M \rightrightarrows B^\ell \underset{A^\ell}{\otimes} B^\ell \underset{A^\ell}{\otimes} M$, $b \otimes m \mapsto b \otimes 1 \otimes m, 1 \otimes b \otimes m$, coincide. Now the exact sequence $0 \to M \to B^\ell \underset{A^\ell}{\otimes} M \to B^\ell \underset{A^\ell}{\otimes} B^\ell \underset{A^\ell}{\otimes} M$ (the latter arrow is the difference of the above two standard ones) together with the left exactness of P^*_A implies the desired descent property. □

For L_i, M as above an *A-polydifferential $*$ operation* is an operation $\varphi \in P^*_I(\{L_i\}, M)$ such that for every finitely generated $A^\ell[\mathcal{D}_X]$-submodules $L'_i \subset L_i$ the corresponding restriction $\in P^*_I(\{L'_i\}, M)$ is an A-polydifferential $*$ operation of finite order (see 1.4.7). The composition of A-polydifferential operations is A-polydifferential, so they define another pseudo-tensor structure on $\mathcal{M}(X, A^\ell)$.

Lemma. A-polydifferential $*$ operations have local nature with respect to the étale topology of SpecA^ℓ.

Proof. The case of arbitrary A-polydifferential operations reduces immediately to that of A-polydifferential operations of finite order, and then to that of A-polylinear ones (replace

L_i by $L_i^{(n)}$ as in 1.4.7 and notice that $L_i^{(n)}$ have étale local nature). Then use the previous lemma. □

Therefore for any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ we have the compound tensor category $\mathcal{M}(\mathcal{Y})^{*!}$ and the sheaf of compound tensor categories $\mathcal{M}(\mathcal{Y}_{\text{ét}})^{*!}$ on $\mathcal{Y}_{\text{ét}}$, the pull-back functors f^* are compound tensor functors, etc. The polydifferential $*$ operations also make sense in this setting.

Remarks. (i) One has the following immediate analog of 2.2.10. Let $\{L_i\}$ be a finite collection of finitely presented $A^\ell[\mathcal{D}_X]$-modules, M any $A^\ell[\mathcal{D}_X]$-module. Then the $A^\ell[\mathcal{D}_X]$-module $\mathcal{P}^*_{A\,I}(\{L_i\}, M)$ is well-defined. The functor $\mathcal{P}^*_{A\,I}$ is compatible with flat pull-backs. So the functor $\mathcal{P}^*_I$ makes sense on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$.

(ii) Thus for every finitely presented $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module L its dual $L' := \mathcal{H}om^*(L, \mathcal{O}_\mathcal{Y}\omega_X)$ is well-defined. In particular, if the $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module of differentials $\Omega_\mathcal{Y} = \Omega_{\mathcal{Y}/X}$ is finitely presented[56] then $\Theta_\mathcal{Y} := \Omega'_\mathcal{Y}$ is well-defined.

(iii) If L is a $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$ then L' is also a $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$, and the canonical pairing is non-degenerate (see 1.4.2).

2.3.13. We are going to discuss the notion of smoothness for algebraic $\mathcal{D}_X$-spaces. First let us recall the definition of formally smooth algebras.

Let $\mathcal{M}$ be any abelian tensor k-category with unit object (so $\otimes$ is right exact). A commutative algebra A in $\mathcal{M}$ is *formally smooth* if for any (commutative) algebra R and an ideal $I \subset R$ such that $I^2 = 0$, any morphism $f: A \to R/I$ may be lifted to a morphism $\tilde{f}: A \to R$. It suffices to check this in the case when f is an isomorphism, i.e., $R/I = A$, $f = id$ (replace arbitrary R, I, f by $\widetilde{R} := A \underset{R/I}{\times} R$, $\widetilde{I} := \mathrm{Ker}(\widetilde{R} \to A) = I$, $f = id$). Note that the symmetric algebra $\mathrm{Sym}V$ is formally smooth if V is a projective object of $\mathcal{M}$.

Assume that $\mathcal{M}$ has many projective objects. We may represent A as a quotient B/J, $B = \mathrm{Sym}\, V$ for a projective V. Then A is formally smooth iff the projection $B/J^2 \to B/J = A$ has a section $\sigma: A \to B/J^2$ (if σ exists, then for any R, I, f as above f lifts to $\hat{f}: B/J^2 \to R$ and one may set $\tilde{f} = f\sigma$). Note that the sections σ correspond

[56] This happens if $\mathcal{Y}$ is (locally) finitely $\mathcal{D}_X$-presented.

bijectively to derivations $\partial: B \to J/J^2$ such that $\partial(x) = x \bmod J^2$ for $x \in J$ (namely, $\sigma(b \bmod J) = b - \partial(b)$). We may consider instead of ∂ the corresponding morphism of B-modules $\varphi: \Omega_B/J \cdot \Omega_B \to J/J^2$; the condition on ∂ means that $\varphi\nu = id_{J/J^2}$ where $\nu: J/J^2 \to \Omega_B/J \cdot \Omega_B$ is the usual map $\nu(x \bmod J^2) = dx \bmod J \cdot \Omega_B$. Therefore $A = B/J$ is formally smooth iff ν is a split injection. Set $\Omega_A^{(-1)} := \mathrm{Ker}\nu$. Since $\mathrm{Coker}\,\nu = \Omega_A$ and $\Omega_B/J \cdot \Omega_B$ is a projective A-module, we see that

A is formally smooth iff $\Omega_A^{(-1)} = 0$ and Ω_A is a projective A-module.

Remark. $\Omega_A^{(-1)}$ is independent of the presentation $A = B/J$; in fact, $\Omega_A^{(-1)}$ is the -1^{th} cohomology of the cotangent complex of A (see [Gr], [Il]).

Let us return to our tensor category $\mathcal{M}^\ell(X)$. Assume for the moment that X is affine; let A^ℓ be a commutative $\mathcal{D}_X$-algebra.

2.3.14 Proposition. (i) A^ℓ is formally smooth iff it is formally smooth as $\mathcal{O}_X$-algebra and Ω_{A^ℓ} is a projective $A^\ell[\mathcal{D}_X]$-module.

(ii) A^ℓ is formally smooth as $\mathcal{O}_X$-algebra iff it is formally smooth as k-algebra.

Proof. The forgetting of $\mathcal{D}_X$-action tensor functor $o^\ell: \mathcal{M}^\ell(X) \to \mathcal{M}_\mathcal{O}(X)$ sends projective objects to projective ones, as well as the corresponding functor $o^\ell_A: \mathcal{M}^\ell(X, A^\ell) \to A^\ell\mathrm{mod}$. Therefore $o^\ell\Omega_{A^\ell} = \Omega_{o^\ell A^\ell}$, $o^\ell\Omega^{(-1)}_{A^\ell} = \Omega^{(-1)}_{o^\ell A^\ell}$, and (i) follows from the above criterion. The statement (ii) follows from the next lemma (the morphism $\Omega_{A^\ell} \to \Omega_X \underset{\mathcal{O}_X}{\otimes} A^\ell$ left inverse to μ corresponds to the canonical derivation $\nabla: A^\ell \to \Omega_X \underset{\mathcal{O}_X}{\otimes} A^\ell$).

2.3.15 Lemma. If A is an $\mathcal{O}_X$-algebra then A is a formally smooth $\mathcal{O}_X$-algebra iff A is formally smooth as k-algebra and the morphism of A-modules $\mu: \Omega_X \underset{\mathcal{O}_X}{\otimes} A \to \Omega_A$ is a split injection.

Proof. Since X is smooth over k, the standard long exact sequence reduces to $0 \to \Omega_A^{(-1)} \to \Omega^{(-1)}_{A/\mathcal{O}_X} \to \Omega_X \underset{\mathcal{O}_X}{\otimes} A \to \Omega_A \to \Omega_{A/\mathcal{O}_X} \to 0$; this implies the lemma. Here is another ad hoc proof. Let us represent A as B/J, $B = \mathrm{Sym}\,V$, V is a free $\mathcal{O}_X$-module. The formal smoothness of A over $\mathcal{O}_X$ means that the morphism of A-modules $\nu: J/J^2 \to \Omega_{B/\mathcal{O}}/J \cdot \Omega_{B/\mathcal{O}}$ admits a left inverse. The formal smoothness over k means that the morphism $\tilde{\nu}: J/J^2 \to$

$\Omega_{B/k}/J\cdot\Omega_{B/k}$, admits a left inverse. It remains to apply to $P = J/J^2$, $Q = \Omega_{B/k}/J\cdot\Omega_{B/k}$, $R = \Omega_X \underset{\mathcal{O}_X}{\otimes} A$ the following

Easy Lemma. Assume we have morphisms $P \xrightarrow{f} Q \xleftarrow{g} R$ where g is a split injection. Then $P \to Q/\mathrm{Im}g$ is a split injection iff such are both $f\colon P \to Q$ and $R \to Q/\mathrm{Im}f$.

Proof. Let us represent Q as $R \oplus S$, so $g(r) = (r, 0)$, $f(p) = (\varphi(p), \psi(p))$. The morphism $R \to Q/f(P)$ is a split injection iff there is $\gamma\colon S \to R$ such that $\varphi = \gamma\psi$, i.e., when $\varphi \in \mathrm{Hom}(S, R)\psi$; the morphism f is a split injection iff $id_P \in \mathrm{Hom}(R, P)\varphi + \mathrm{Hom}(S, P)\psi$. We need to show that $id_P \in \mathrm{Hom}(S, P)\psi$ iff $id_P \in \mathrm{Hom}(R, P)\varphi + \mathrm{Hom}(S, P)\psi$ and $\varphi \in \mathrm{Hom}(S, R)\psi$, which is clear. $\square$

2.3.16. ¿From now on we do *not* assume that X is affine. For a $\mathcal{D}_X$-algebra A^ℓ the $A^\ell[\mathcal{D}_X]$-modules $\Omega_{A^\ell}, \Omega_{A^\ell}^{(-1)}$ are compatible with étale localization of A^ℓ. Therefore for any $\mathcal{D}_X$-scheme, or algebraic $\mathcal{D}_X$-space, $\mathcal{Y}$ we have $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules $\Omega_\mathcal{Y}$, $\Omega_\mathcal{Y}^{(-1)}$. We say that $\mathcal{Y}$ is *formally smooth* if $\Omega_\mathcal{Y}^{(-1)} = 0$ and $\Omega_\mathcal{Y}$ is a locally projective $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module. In fact, $\mathcal{Y}$ is formally smooth if and only if for every affine scheme S equipped with a $\mathcal{D}_X$-scheme structure and every closed $\mathcal{D}_X$-subscheme $S_0 \subset S$ defined by an ideal I with $I^2 = 0$ the map $\mathrm{Mor}_{\mathcal{D}_X}(S, \mathcal{Y}) \to \mathrm{Mor}_{\mathcal{D}_X}(S_0, \mathcal{Y})$ is surjective. To prove the "if" statement notice that locally a $\mathcal{D}_X$-morphism $f : S_0 \to \mathcal{Y}$ can be extended to a morphism $S \to \mathcal{Y}$ and these extensions form a torsor over $\mathcal{H}om(f^*\Omega_\mathcal{Y}, I)$. Such a torsor is the same as an extension of $f^*\Omega_\mathcal{Y}$ by I. Now $f^*\Omega_\mathcal{Y}$ is locally projective and therefore projective (see 2.3.7), so $\mathrm{Ext}(f^*\Omega_\mathcal{Y}, I) = 0$ and our torsor is trivial.

We say that $\mathcal{Y}$ is *smooth* if it is formally smooth and $\Omega_\mathcal{Y}$ is a finitely generated $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module. Then $\Omega_\mathcal{Y}$ is a $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$.[57]

Remark. One may argue that the term "smooth" should be reserved for formally smooth $\mathcal{D}_X$-schemes which have locally $\mathcal{D}_X$-finite type (i.e., have a covering by $\mathrm{Spec}\, A^\ell$ where A^ℓ is a finitely generated $\mathcal{D}_X$-algebra), or $\mathcal{D}_X$-finite presentation. These conditions are a priori stronger than the one we consider above. We do not know if a smooth $\mathcal{D}_X$-scheme of $\mathcal{D}_X$-finite type is necessarily $\mathcal{D}_X$-finitely presented.

[57] In fact, the latter property is the only property of smoothness that will be actually used in this book.

2.3.17. A morphism $\varphi : \mathcal{Y} \to \mathcal{Z}$ of algebraic $\mathcal{D}_X$-spaces is *formally smooth* if for every (S, S_0) as above and any pair of φ-compatible morphisms $f : S_0 \to \mathcal{Y}$, $g : S \to \mathcal{Z}$ lifts to a morphism $\tilde{f} : S \to \mathcal{Y}$. The reformulation of this property in terms of relative differentials is left to the reader. One says that φ is *formaly étale* if, in addition, $\tilde{f}$ is unique.

2.3.18. For a smooth $\mathcal{D}_X$-scheme $\mathcal{Y}$ a *coordinate system* on $\mathcal{Y}$ is a morphism $\nu\colon \mathcal{Y} \to \operatorname{Spec} \operatorname{Sym}^{\cdot}(\mathcal{D}_X^n)$, $n \geq 0$, such that $d\nu\colon \nu^*\Omega_{\operatorname{Sym}^{\cdot}(\mathcal{D}_X^n)} \to \Omega_{\mathcal{Y}}$ is an isomorphism. The jet schemes $\jmath Z$, Z/X is smooth, have local coordinate systems (for any étale morphism $\varphi\colon Z \to \mathbb{A}_X^n$ the morphism $\jmath\varphi\colon \jmath Z \to \jmath\mathbb{A}_X^n = \operatorname{Spec} \operatorname{Sym}^{\cdot}(\mathcal{D}_X^n)$ is a coordinate system on $\jmath Z$). This is **not** true (even at a generic point!) for arbitrary smooth $\mathcal{D}_X$-schemes.

Here is the reason. Assume that our $\mathcal{Y} = \operatorname{Spec} A^\ell$ is a smooth reduced[58] $\mathcal{D}_X$-scheme such that $\Omega_{A^\ell} \simeq A^\ell[\mathcal{D}_X]$. Note that any invertible element of $A^\ell[\mathcal{D}_X]$ lies in $A^\ell \subset A^\ell[\mathcal{D}_X]$. Denote by $\mathcal{L}$ the image of A^ℓ by any isomorhism $A^\ell[\mathcal{D}_X] \xrightarrow{\sim} \Omega_{A^\ell}$. This is a canonically defined line subbundle of Ω_A. If $\mathcal{Y}$ admits locally a coordinate system, then $\mathcal{L}$ is an integrable subbundle, i.e., $d(\mathcal{L}) \subset \mathcal{L} \wedge \Omega_{A^\ell} \subset \Omega^2_{A^\ell}$.

2.3.19 Example. Let us construct a smooth A^ℓ with non-integrable $\mathcal{L}$. Take $X = \operatorname{Spec} k[t]$, $B^\ell = k[t, u, v, u', v', \ldots, v'^{-1}]$ (here $u' := \partial_t u$, etc., so B^ℓ is the localization of $\operatorname{Sym}^{\cdot}(\mathcal{D}_X^2) = k[t, u, v, u', \ldots]$ by v'). Let $I \subset B^\ell$ be the $\mathcal{D}_X$-ideal generated by $\varphi = v + u'v'$; set $A^\ell := B^\ell/I$. Consider the morphism of $B^\ell[\mathcal{D}_X]$-modules

$$\chi\colon B^\ell[\mathcal{D}_X] \oplus B^\ell[\mathcal{D}_X] \to \Omega_{B^\ell}, \quad (\partial_1, \partial_2) \mapsto \partial_1 d\varphi + \partial_2 \xi,$$

where $\xi := du + (u'/v')dv \in \Omega_{B^\ell}$. Localizing B^ℓ (and A^ℓ) further by $v' - u''v' + u'v''$, we see that χ becomes an isomorphism. Now A^ℓ is smooth, and the map $A^\ell[\mathcal{D}_X] \to \Omega_{A^\ell}$, $\partial \mapsto \partial\xi_{A^\ell}$, is an isomorphism. Here ξ_{A^ℓ} is restriction of ξ to $\operatorname{Spec} A^\ell$; this is a generator of $\mathcal{L}$. One has $\xi_{A^\ell} \wedge d\xi_{A^\ell} = \xi_{A^\ell} \wedge d(u'/v') \wedge dv = -\xi_{A^\ell} \wedge d(v/v'^2) \wedge dv = \alpha\xi_{A^\ell} \wedge dv \wedge dv'$, $\alpha \neq 0$, where u and v are considered as elements of A^ℓ. It is easy to check that $dv = \beta\partial_t\xi_{A^\ell}$, $\beta \neq 0$, so $\xi_{A^\ell} \wedge d\xi_{A^\ell} \neq 0$ and $\mathcal{L}$ is non-integrable.

[58] We do not know if a smooth $\mathcal{D}_X$-scheme is necessarily reduced.

§4 The spaces of horizontal sections

In this section we assume that $X \neq \emptyset$ is connected of dimension n.

2.4.1. Denote by $\mathcal{C}omu(k)$ the category of commutative unital k-algebras. For any $F \in \mathcal{C}omu(k)$ the constant left $\mathcal{D}$-module $F \otimes \mathcal{O}_X$ is a $\mathcal{D}_X$-algebra in the obvious way. This functor $\mathcal{C}omu(k) \to \mathcal{C}omu_{\mathcal{D}}(X)$ identifies $\mathcal{C}omu(k)$ with the full subcategory of constant $\mathcal{D}_X$-algebras.[59] It has a right adjoint functor $\Gamma^{\nabla}\colon \mathcal{C}omu_{\mathcal{D}}(X) \to \mathcal{C}omu(k)$, $\Gamma^{\nabla}(A^{\ell}) = \Gamma^{\nabla}(X, A^{\ell}) = \mathrm{Hom}(\mathcal{O}_X, A^{\ell}) = H^{-n}_{DR}(X, A)$.

We would like to find out if our embedding admits a left adjoint functor

$$\mathcal{C}omu_{\mathcal{D}}(X) \to \mathcal{C}omu(k), \quad A^{\ell} \mapsto \langle A \rangle = \langle A \rangle(X). \tag{2.4.1.1}$$

So we want to see if for $A^{\ell} \in \mathcal{C}omu_{\mathcal{D}}(X)$ the functor $\langle A \rangle\colon F \mapsto \mathrm{Hom}(A^{\ell}, F \otimes \mathcal{O}_X)$ on $\mathcal{C}omu(k)$ is representable. Notice that $\mathrm{Spec}\,\langle A \rangle$ is the space of horizontal sections $X \to \mathrm{Spec}\, A^{\ell}$.

It is clear that $\langle A \rangle$ is a projective limit of commutative algebras. Namely, consider the set $\mathcal{C}$ of ideals $I \subset A^{\ell}$ such that A^{ℓ}/I is a constant $\mathcal{D}$-algebra. The intersection of two ideals from $\mathcal{C}$ belongs to $\mathcal{C}$, so it is a subfilter in the ordered set of all ideals in A^{ℓ}. For any $I \in \mathcal{C}$ we have a commutative algebra $\Gamma^{\nabla}(X, A^{\ell}/I)$, and for $I' \supset I$ the surjective map $\Gamma^{\nabla}(X, A^{\ell}/I) \twoheadrightarrow \Gamma^{\nabla}(X, A^{\ell}/I')$. Our $\langle A \rangle$ is the $\mathcal{C}$-projective limit of these algebras.

Notice that $\langle A \rangle(X)$ is covariantly functorial with respect to open embeddings of X's. Precisely, for any open $X' \subset X$ the algebra $\langle A \rangle(X)$ is a quotient algebra of $\langle A \rangle(X')$ in the obvious way.

For a point $x \in X$ the fiber[60] $A^{\ell}_x := H^0 i_x^* A^{\ell} = A^{\ell} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_x/\mathrm{m}_x$ of A^{ℓ} at x is a commutative algebra. A $\mathcal{D}_X$-algebra epimorphism $A^{\ell} \to F \otimes \mathcal{O}_X$, $F \in \mathcal{C}omu(k)$, induces an epimorphism $A^{\ell}_x \to (F \otimes \mathcal{O}_X)_x = F$, so we get a canonical morphism $A^{\ell}_x \to \langle A \rangle$ with dense image. Therefore $\mathrm{Spec}\,\langle A \rangle$ is naturally an ind-subscheme of any fiber $\mathrm{Spec}\, A^{\ell}_x$.

2.4.2 Proposition. If X is compact, then $\langle A \rangle$ is a true commutative algebra.

[59] i.e., those $\mathcal{D}_X$-algebras which are constant as a $\mathcal{D}_X$-module, see 2.1.12.

[60] Here i_x is the embedding $\{x\} \hookrightarrow X$.

Proof. We want to show that A admits the maximal constant $\mathcal{D}$-algebra quotient. Consider the maximal constant $\mathcal{D}$-module quotient $\sigma_A : A \twoheadrightarrow A_{const}$ (see 2.1.12). Now take the quotient of A modulo the ideal generated by the kernel of σ_A. $\square$

We saw in 2.1.12 that $A^\ell_{const} = H^n_{DR}(X, A) \otimes \mathcal{O}_X$, so $\langle A \rangle$ is a quotient of $H^n_{DR}(X, A)$. In 2.4.3–2.4.5 below we give some "explicit" descriptions of $\langle A \rangle$. We assume that X is compact.

Let $\mathcal{I}^\ell \subset A^\ell \otimes A^\ell$ be the kernel of the multiplication map $A^\ell \otimes A^\ell \to A^\ell$. Consider the morphism $\sigma_A^{\otimes 2} : A^\ell \otimes A^\ell \to H^n_{DR}(X, A)^{\otimes 2} \otimes \mathcal{O}_X$. Restricting it to $\mathcal{I}$ and passing to de Rham cohomology we get a morphism

$$\nu : H^n_{DR}(X, \mathcal{I}) \to H^n_{DR}(X, A)^{\otimes 2}. \tag{2.4.2.1}$$

The projection $H^n_{DR}(X, A) \twoheadrightarrow \langle A \rangle$ together with the product on $\langle A \rangle$ identifies $\langle A \rangle$ with a quotient of $H^n_{DR}(X, A)^{\otimes 2}$.

2.4.3 Lemma. One has $\langle A \rangle = \mathrm{Coker}\nu$.

Proof. Look at the commutative diagram

$$\begin{array}{ccc} A^\ell & \longrightarrow & \langle A \rangle \otimes \mathcal{O}_X \\ \uparrow & & \uparrow \\ A^\ell \otimes A^\ell & \longrightarrow & H^n_{DR}(X, A)^{\otimes 2} \otimes \mathcal{O}_X \end{array}$$

According to the proof of 2.4.2, $\langle A \rangle \otimes \mathcal{O}_X$ is the quotient of A^ℓ modulo the ideal $A^\ell \cdot \mathrm{Ker}\sigma_A$, or, equivalently, the quotient of $A^\ell \otimes A^\ell$ modulo $\mathcal{I} + \mathrm{Ker}\sigma_A \otimes A^\ell + A^\ell \otimes \mathrm{Ker}\sigma_A$. This is $\mathrm{Coker}\nu \otimes \mathcal{O}_X$, q.e.d. $\square$

Here is another description of $\langle A \rangle$. Let $\Delta\colon X \hookrightarrow X \times X$ be the diagonal embedding, $j\colon U := X \times X \setminus \Delta(X) \hookrightarrow X \times X$ its complement. Consider the morphism $j_* j^* A \boxtimes A \to \Delta_* A[-n+1]$ in $D^b\mathcal{M}(X \times X)$ defined as the composition of the canonical "residue" morphism $j_* j^*(A \boxtimes A) \to \Delta_* \Delta^!(A \boxtimes A)[1] \xrightarrow{\sim} \Delta_*(A \overset{L}{\otimes}{}^! A)[-n+1]$ and the product $A \overset{L}{\otimes}{}^! A \to A \otimes^! A \to A$. Passing to de Rham cohomology[61] we get a morphism of vector spaces

$$\mu : H^{2n-1}_{DR}(U, A \boxtimes A) \to H^n_{DR}(X, A). \tag{2.4.3.1}$$

[61] One has $H^\cdot_{DR}(X \times X, j_* j^* B) = H^\cdot_{DR}(U, B)$, $H^\cdot_{DR}(X \times X, \Delta_* C) = H^\cdot_{DR}(X, C)$.

2.4.4 Lemma. One has $\langle A\rangle = \mathrm{Coker}\mu$.

Proof. Denote by C the cone of our morphism $j_*j^*A \boxtimes A[-1] \to \Delta_* A[-n]$; this is an object of $D^b\mathcal{M}(X\times X)$.[62] One has[63] $\mathrm{Coker}\mu = H^{2n}_{DR}(X\times X, C)$.

Consider the obvious morphism $A\boxtimes A \to C$. Denote by D its cone, so we have the exact triangle $A\boxtimes A \to C \to D$. The complex D is supported on the diagonal; its top non-zero cohomology is $H^{n-1}D = \Delta_*\mathcal{J}$. Therefore $H^{2n-1}_{DR}(X\times X, D) = H^n_{DR}(X,\mathcal{J})$ and all the higher de Rham cohomology vanish. So the long exact sequence of de Rham cohomology identifies $H^{2n}_{DR}(X\times X, C)$ with the cokernel of the arrow $H^{2n-1}_{DR}(X\times X, D) \to H^{2n}_{DR}(X\times X, A\boxtimes A)$ which may be rewritten as $\nu' : H^n_{DR}(X,\mathcal{J}) \to H^n_{DR}(X,A)^{\otimes 2}$.

So we see that $\mathrm{Coker}\mu = \mathrm{Coker}\nu'$. We leave it to the reader to check that our ν' equals the map ν from (2.4.2.1). Now use 2.4.3. □

Now let $S \subset X$ be a finite non-empty subset. We have the canonical residue map $\mathrm{Res}_S = \Sigma \mathrm{Res}_s \colon H^{n-1}_{DR}(X\smallsetminus S, A) \to \underset{s\in S}{\oplus} A^\ell_s$ (see 2.1.12); denote its image by V_S. Consider the product $\underset{S}{\otimes} A^\ell_s \twoheadrightarrow \langle A\rangle$ of our surjections $A^\ell_s \twoheadrightarrow \langle A\rangle$ (see 2.4.1, 2.4.2). As follows from the definition of $\langle A\rangle$ (and 2.1.12), its kernel contains $f(V_S)$ where $f : \underset{S}{\oplus} A^\ell_s \to \underset{S}{\otimes} A^\ell_s$ maps $a_s \in A^\ell_s$ to $a_s \otimes 1^{\otimes S\smallsetminus\{s\}}$. Hence it contains the ideal I_S generated by $f(V_S)$. So we have a ring epimorphism $\alpha_S : (\underset{S}{\otimes} A^\ell_s)/I_S \to \langle A\rangle$.

2.4.5 Lemma. One has $\alpha_S : (\underset{S}{\otimes} A^\ell_s)/I_S \xrightarrow{\sim} \langle A\rangle$.

Proof. Consider the canonical projections $\pi_s : A^\ell_s \twoheadrightarrow (A^\ell_{const})_s = H^n_{DR}(X,A)$. According to 2.1.12 the kernel of $\pi_S = \Sigma\pi_s : \underset{S}{\oplus} A^\ell_s \twoheadrightarrow H^n_{DR}(X,A)$ equals to V_S.

(a) Assume that S consists of a single point $s \in X$. As follows from the proof of 2.4.2 $\langle A\rangle \otimes \mathcal{O}_X$ is the quotient of A^ℓ modulo the ideal generated by the kernel of the projection $\pi : A^\ell \twoheadrightarrow A^\ell_{const}$. Therefore, passing to the s-fibers, we see that $\langle A\rangle$ is the quotient of A^ℓ_s modulo the ideal generated by $\mathrm{Ker}\pi_s$, i.e., by V_s, and we are done.

(b) Assume that $|S| \geq 2$. Take any $s_0 \in S$. Then $A^\ell_{s_0} \cap V_S = V_{s_0}$ (indeed, both subspaces are equal to the kernel of π_{s_0}). Therefore the morphism $A^\ell_{s_0} \to \underset{S}{\otimes} A^\ell_s$ yields a morphism

[62] In the most important case $n = 1$ our C is actually a $\mathcal{D}$-module, not merely a complex.

[63] Since U is non-compact $H^{2n}_{DR}(U,B) = 0$ for every $B \in \mathcal{M}(U)$.

$\zeta : A^\ell_{s_0}/I_{s_0} \to (\underset{S}{\otimes} A^\ell_s)/I_S$. Since $A^\ell_{s_0}+V_S = \underset{S}{\oplus} A^\ell_s$ (which follows from the surjectivity of π_{s_0}) we see that ζ is surjective. The composition of ζ with α_S is the map $\alpha_{s_0} : A^\ell_{s_0}/I_{s_0} \to \langle A\rangle$ for the one point set $\{s_0\}$ which we know to be an isomorphism by (a). Thus α_S is an isomorphism. □

Remark. One may also deduce 2.4.5 in the case $S = \{s\}$ (i.e., (a) above) from 2.4.4. Namely, set $U_s := X \smallsetminus \{s\}$; denote by i_s, $\tilde{i}_s$ the embeddings $\{s\} \hookrightarrow X$, $X = \{s\} \times X \hookrightarrow X \times X$. Then $\tilde{i}^!_s(A \boxtimes A) = (i^!_s A) \otimes A$, so we have a commutative diagram

$$\begin{array}{ccc} H^{2n-1}_{DR}(U, A \boxtimes A) & \xrightarrow{\mu} & H^n_{DR}(X, A) \\ \uparrow & & \uparrow \\ A^\ell_s \otimes H^{n-1}_{DR}(U_s, A) & \xrightarrow{\mu_s} & A^\ell_s \end{array}$$

the vertical arrows come from canonical morphisms $\tilde{i}_{s*}\tilde{i}^!_s(A \boxtimes A) \to A \boxtimes A$, $i_{s*}i^!_s A \to A$, and μ_s sends $a \otimes b$ to $a\,\mathrm{Res}_s b$. The standard long exact sequences show that the vertical arrows are surjective; the kernel of the right one is $\mathrm{Im}\,(\mathrm{Res}_s)$. Since $\mathrm{Res}_s(b) = \mu_s(1 \otimes b)$, one has $\mathrm{Im}\,(\mathrm{Res}_s) \subset \mathrm{Im}(\mu_s)$. Therefore, $\mathrm{Coker}\mu = A^\ell_s/\mathrm{Im}(\mu_s) = A^\ell_s/I_s$, so, by 2.4.4, we have $A^\ell_s/I_s \xrightarrow{\sim} \langle A\rangle$. One checks immediately that this is the morphism of 2.4.5.

2.4.6. Let $j : U \hookrightarrow X$ be a non-empty Zariski open subset (we assume that X is compact). For $A^\ell \in \mathcal{C}om u_{\mathcal{D}}(U)$ the left $\mathcal{D}$-module $j.A^\ell := H^0 j_* A^\ell$ is a $\mathcal{D}_X$-algebra in the obvious way. Let $\Xi^c(A)$ denotes the set of $\mathcal{D}_X$-subalgebras $A^\ell_\xi \subset j.A^\ell$ such that $j^* A^\ell_\xi = A^\ell$. The intersection of two subalgebras from $\Xi^c(A)$ belongs to $\Xi^c(A)$, so this is a subfilter in the ordered set of all $\mathcal{D}_X$-subalgebras of $j.A^\ell$. We have a $\Xi^c(A)$-projective system of algebras $\langle A_\xi\rangle(X)$ connected by epimorphisms. The canonical maps $\langle A\rangle(U) \to \langle A_\xi\rangle(X)$ yield a morphism (the limit is taken over $\Xi^c(A)$)

$$\langle A\rangle(U) \to \varprojlim \langle A_\xi\rangle(X). \tag{2.4.6.1}$$

2.4.7 Lemma. This is an isomorphism.

Proof. It suffices to show that any morphism[64] $\varphi\colon A^\ell \to F \otimes \mathcal{O}_U$ is the restriction to U of some morphism $\varphi_X\colon A^\ell_\xi \to F \otimes \mathcal{O}_X$, $A^\ell_\xi \in \Xi^c(A)$. Note that φ defines a morphism of

[64] Here F is a commutative algebra.

$\mathcal{D}_X$-algebras $j.(\varphi)$: $j.A^\ell \to j.(F \otimes \mathcal{O}_U)$, and $F \otimes \mathcal{O}_X$ is a $\mathcal{D}_X$-subalgebra of $j.(F \otimes \mathcal{O}_U)$. Set $A^\ell_\xi := j.(\varphi)^{-1}(F \otimes \mathcal{O}_X)$, $\varphi_X := j.(\varphi)\big|_{A^\ell_\xi}$. We are done. $\square$

2.4.8. For the rest of the section our X is a curve. Let $x \in X$ be a (closed) point, $j_x : U_x \hookrightarrow X$ the complement to x. For $A^\ell \in \mathcal{C}omu_{\mathcal{D}}(U_x)$ consider the topology $\Xi^{as}_x = \Xi^{as}_x(j_{x*}A)$ on $j_{x*}A$ at x (see 2.1.13 for terminology) formed by all $\mathcal{D}_X$-subalgebras A^ℓ_ξ such that $j_x^* A_\xi = A$. We have

$$A^\ell_{\xi x} = i_x^!((j_{x*}A)/A_\xi) = h((j_{x*}A)/A_\xi)_x. \tag{2.4.8.1}$$

We see that for $A^\ell_{\xi'} \subset A^\ell_\xi$ the morphism of fibers $A^\ell_{\xi' x} \to A^\ell_{\xi x}$ is surjective. Denote by A^{as}_x the Ξ^{as}_x-projective limit of the commutative algebras $A^\ell_{\xi x}$; as a plain topological vector space it equals to the Ξ^{as}_x-completion of the vector space $h(j_{x*}A)_x$ (see 2.1.14).[65]

Remark. For an A^ℓ-module M let $\Xi^A_x(j_{x*}M)$ be the topology at x[66] on $j_{x*}M$ formed by $M_\xi \in \Xi_x(j_{x*}M)$ such that M_ξ is an A^ℓ_ξ-submodule of $j_{x*}M$ for some $A_\xi \in \Xi^{as}_x(j_{x*}A)$. Then the completion $\hat{h}^A_x(j_{x*}M)$ is a topological A^{as}_x-module.[67]

Let us show that the ind-affine ind-scheme $\operatorname{Spf} A^{as}_x := \varinjlim \operatorname{Spec} A^\ell_{\xi x}$, $\xi \in \Xi^{as}_x$, is the space of horizontal sections of $\operatorname{Spec} A^\ell$ over the formal punctured disc at x. Denote by K_x the field of fractions of $O_x := \varprojlim \mathcal{O}_x/\mathrm{m}_x^n$. For a commutative algebra F set $O_x \mathbin{\widehat{\otimes}} F := \varprojlim (O_x/\mathrm{m}_x^n) \otimes F$, $K_x \mathbin{\widehat{\otimes}} F := K_x \underset{O_x}{\otimes} (O_x \mathbin{\widehat{\otimes}} F)$.[68] We consider $O_x \mathbin{\widehat{\otimes}} F \subset K_x \mathbin{\widehat{\otimes}} F$ as sheaves on X supported at x; these are *non-quasicoherent* commutative $\mathcal{D}_X$-algebras in the obvious way.

2.4.9 Lemma. There is a canonical identification

$$\operatorname{Hom}(A^{as}_x, F) = \operatorname{Hom}(j_{x*}A^\ell, K_x \mathbin{\widehat{\otimes}} F). \tag{2.4.9.1}$$

Here the former Hom denotes the set of continuous morphisms of commutative algebras, the latter one that of commutative $\mathcal{D}_{U_x}$-algebras.

[65] The superscript "as" means "associative"; this notation will be clearified in 3.4.

[66] See 2.1.13.

[67] Since $h(j_{x*}M/M_\xi)$ is an $A^\ell_{\xi x}$-module.

[68] If t is a parameter at x then $O_x \mathbin{\widehat{\otimes}} F = F[[t]]$, $K_x \mathbin{\widehat{\otimes}} F = F((t))$.

Proof (cf. the proof of 2.4.7). The map $\mathrm{Hom}(j_{x*}A^\ell, K_x \mathbin{\widehat{\otimes}} F) \to \mathrm{Hom}(A_x^{as}, F)$ is defined as follows. Take $\varphi: j_{x*}A^\ell \to K_x \mathbin{\widehat{\otimes}} F$. Set $A^\ell_\varphi := \varphi^{-1}(O_x \mathbin{\widehat{\otimes}} F) \subset j_{x*}A^\ell$. Since $K_x \mathbin{\widehat{\otimes}} F/O_x \mathbin{\widehat{\otimes}} F$ is a quasicoherent $\mathcal{O}_X$-module supported at x we see that $A^\ell_\varphi \in \Xi^{ch}_x$. We have the morhism of $\mathcal{D}_X$-algebras $\varphi: A^\ell_\varphi \to O_x \mathbin{\widehat{\otimes}} F$. Let $\varphi_x: (A^\ell_\varphi)_x \to F$ be the morphism of fibers at x.

We leave it to the reader to check that the map $\varphi \mapsto (A^\ell_\varphi, \varphi_x) \in \mathrm{Hom}(A_x^{as}, F)$ is bijective. □

Remark. The above lemma is essentially a particular case of a general statement about $\mathcal{D}$-modules: see Remark after Lemma 3.5.4.

2.4.10. Let $S \subset X$ be a finite set, $j_S : U_S \hookrightarrow X$ its complement. For a $\mathcal{D}$-algebra A on U_S set $A^{as}_S = \widehat{\bigotimes_{s\in S}} A^{as}_s := \varprojlim \bigotimes_{s\in S} A^\ell_{\xi_s s}$. Assume now that X is compact. Subalgebras from $\Xi^c(A)$ (see 2.4.6) are just intersections of subalgebras from $\Xi^{as}_s(j_{S*}A)$, $s \in S$, so, by 2.4.7, we have the canonical maps $A^{as}_s \to \langle A\rangle(U_S)$. They define a continuous morphism

$$v: A^{as}_S \to \langle A\rangle(U_S). \tag{2.4.10.1}$$

By 2.4.5 v identifies $\langle A\rangle(U_S)$ with the quotient of A^{as}_S modulo the closed ideal generated by the image of the morphism

$$r_S = \varprojlim \mathrm{Res}^{A_\xi}_S : \ H^0_{DR}(U_S, A) \to \bigoplus_S A^{as}_s \to A^{as}_S. \tag{2.4.10.2}$$

Remark. r_S obviously factors through the quotient $\Gamma(U_S, h(A))$ of $H^0_{DR}(U_S, A)$.

§5 Lie* ALGEBRAS AND ALGEBROIDS

The notion (if not the name) of Lie* algebra has been, undoubtedly, well-known in mathematical physics for a long time. Since the first version of the present article appeared as a preprint, Lie* algebras migrated to mathematical literature, e.g. they present in [K] (under the alias "conformal algebras") and in [P] (as "vertex Lie algebras").[69]

For a list of basic examples see 2.5.7–2.5.10; an important example of Tate's extension is treated separately in the next section.

[69] The article [P] also refers to a preprint of Dong, Li, and Mason (which we were not able to locate) for "a more general notion of local vertex Lie algebra".

2.5.1. Having at hand the $*$ pseudo-tensor structure, we may consider for any k-operad $\mathcal{B}$ the category $\mathcal{B}^*(X)$ of $\mathcal{B}$ algebras in $\mathcal{M}(X)^*$, and for any $L \in \mathcal{B}^*(X)$ (and a $\mathcal{B}$-module operad $\mathcal{C}$) the corresponding category $\mathcal{M}(X,L) = \mathcal{M}(X)(L,\mathcal{C})$ of L-modules. We have the corresponding sheaves of categories $\mathcal{B}^*(X_{\text{ét}})$ and $\mathcal{M}(X_{\text{ét}},L)$ on $X_{\text{ét}}$.

For a locally closed embedding $i\colon X \hookrightarrow Y$ we have the adjoint functors $\mathcal{B}^*(X) \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{B}^*(Y)$ (see 2.2.5). If X is a closed subscheme then i_* identifies $\mathcal{B}^*(X)$ with the full subcategory of $\mathcal{B}^*(Y)$ that consists of algebras supported on X.

2.5.2. The functor h sends $\mathcal{B}^*$ algebras on X to $\mathcal{B}$ algebras in the tensor category $Sh^{\otimes}(X)$ of sheaves of k-vector spaces on X. For $L \in \mathcal{B}^*(X)$ an L-module structure on a $\mathcal{D}$-module M defines on M the $\mathcal{C}(h(L))$-module structure (see 1.2.17). According to 2.2.16 the functor

$$\mathcal{M}(X,L) \to \mathcal{C}(h(L)) - \text{modules in } \mathcal{M}(X) \tag{2.5.2.1}$$

is a fully faithful embedding.

2.5.3. We will be mainly interested in Lie* algebras. Therefore a Lie* algebra is a $\mathcal{D}$-module L equipped with a Lie* bracket, i.e., a $*$-pairing $[\ \ ] \in P_2^*(\{L,L\},L)$ which is skew-symmetric and satisfies the Jacobi identity. Then the sheaf $h(L)$ is a Lie algebra with bracket $[\bar{\ell}_1,\bar{\ell}_2] = \overline{[\ \ ](\ell_1 \boxtimes \ell_2)}$. Recall that $\mathcal{C}(h(L))$ is the universal enveloping algebra of $h(L)$ and a $\mathcal{C}(h(L))$-module is the same as an $h(L)$-module.

2.5.4. An L-module structure on $M \in \mathcal{M}(X)$ yields an $h(L)$-action on M given by

$$\bar{\ell}(m) = Tr_2(\ \cdot(\ell \boxtimes m)). \tag{2.5.4.1}$$

Here $\cdot \in P_2^*(\{L,M\},M)$ is the L-action and $Tr_2\colon \Delta^{\cdot}\Delta_* M = pr_{2\cdot}\Delta_* M \to M$ is the trace map for the projection $pr_2\colon X \times X \to X$.

According to 2.2.16 this way we get a one-to-one correspondence between the L-module structures on $M \in \mathcal{M}(X)$ and such $h(L)$-actions on M (we call them *good* actions) that for every $m \in M$ the map $L \to M$, $\ell \mapsto \bar{\ell}m$, is a differential operator.

Remark. For an L-module M and a section $\bar{m}$ of $h(M)$ the stabilizer $L^{\bar{m}}$ of $\bar{m}$ (defined as the kernel of the morphism $L \to M$, $l \mapsto l\bar{m}$) is a Lie* subalgebra of L.

Examples. (i) Let V be a coherent $\mathcal{D}$-module. Then the $h(\mathcal{E}nd^*V)$-action on V coincides with the map $h(\mathcal{H}om(V, V \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X)) \to \mathrm{End}V$, which sends $\bar{\varphi}$, $\varphi \in \mathcal{H}om(V, V \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X)$, to the composition $V \xrightarrow{\varphi} V \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to V$, the second arrow is $v \otimes \partial \mapsto v\partial$

(ii) The adjoint action of L yields an $h(L)$-action on L denoted by $\bar{\ell} \mapsto ad_{\bar{\ell}} \in \mathrm{End}L$.

2.5.5 Proposition. For any $\mathcal{D}$-module L the Lie* brackets on L are in $1-1$ correspondence with morphism of sheaves $h(L) \to \mathrm{End}L$, $\bar{\ell} \mapsto ad_{\bar{\ell}}$, which satisfy the following properties (below we set $ad_{\bar{\ell}}(\bar{\ell}') := \overline{ad_{\bar{\ell}}(\ell')}$):

(i) $\left[ad_{\bar{\ell}_1}, ad_{\bar{\ell}_2}\right] = ad_{ad_{\bar{\ell}_1}(\bar{\ell}_2)}$ for any $\bar{\ell}_1, \bar{\ell}_2 \in h(L)$,

(ii) $ad_{\bar{\ell}}(\ell) = 0$ for any $\ell \in L$,

(iii) For any $\ell' \in L$ the map $L \to L$, $\ell \mapsto ad_{\bar{\ell}}(\ell')$, is a differential operator (with respect to the $\mathcal{O}$-module structure on L).

Proof. We use 2.2.16. The condition (iii) is exactly the condition $(*)$ of 2.2.16, hence it means that ad comes from a $*$-pairing $[\ \ ] \in P_2^*(\{L, L\}, L)$. The property (i) is equivalent to the Jacobi identity by 2.2.16. The property (ii) means that $Tr_2[\ \ ]: L^{\otimes 2} \to L$ vanishes on the symmetric part $L^{\otimes 2+} \subset L^{\otimes 2}$. Set $N := \Delta_* L$. Since $[\ \ ]: L^{\otimes 2} \to N$ is $\mathcal{D}_X^{\otimes 2}$-linear $M := [\ \ ](L^{\otimes 2+})$ is a $\mathcal{D}_X^{\otimes 2+}$-submodule of N such that $Tr_2(M) = 0$. So the skew-symmetry of $[\ \ ]$ follows from the next lemma:

2.5.6 Lemma. If $M \subset N$ is a $\mathcal{D}_X^{\otimes 2+}$-submodule such that $Tr_2(M) = 0$ then $M = 0$.

Proof. Take $m \in M$. For any$f \in \mathcal{O}_X$, $\theta \in \Theta_X$ we have $Tr_2(m \cdot (f \otimes \theta + \theta \otimes f)) = 0$. Since $Tr_2(n \cdot (\theta \otimes 1)) = 0$ and $Tr_2(n \cdot (1 \otimes \theta)) = Tr_2(n) \cdot \theta$ for any $n \in N$ we obtain $(Tr_2(m \cdot (f \otimes 1)))\theta = 0$. Since $\Theta_X \mathcal{D}_X = \mathcal{D}_X$ we have$Tr_2(m \cdot (f \otimes 1)) = 0$ for any $f \in \mathcal{O}_X$. Choose a coordinate system $t_1, \ldots, t_n$ on$X$ and for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^k$ set $t^\alpha := t_1^{\alpha_1} \ldots t_k^{\alpha_k} \in \mathcal{O}_X$, $\partial^\alpha := \partial_{t_1}^{\alpha_1} \ldots \partial_{t_k}^{\alpha_k} \in \mathcal{D}_X$. Since $N = L \cdot (\mathcal{D}_X \otimes 1)$ we can write m as $\sum_\alpha \ell_\alpha \cdot (\partial^\alpha \otimes 1)$, $\ell_\alpha \in L$. Then $Tr_2(m \cdot (f \otimes 1)) = \sum_\alpha \ell_\alpha \partial^\alpha(f)$. The equality $\sum_\alpha \ell_\alpha \partial^\alpha(f) = 0$ for all $f \in \mathcal{O}_X$ implies $\ell_\alpha = 0$ (set $f = t^\beta$). □

Here are some basic examples of Lie* algebras:

2.5.7. (a) For any coherent $\mathcal{D}_X$-module M we have an associative*, hence Lie*, algebra

$\mathcal{E}nd^*(M)$ (see 2.2.12, 2.2.13). If M is a $\mathcal{D}_X$-vector bundle then $\mathcal{E}nd^*(M)$ considered as a Lie* algebra is denoted by $\mathfrak{g}l(M)$.

(b) By 2.2.4 (i) for an $\mathcal{O}$-module P a Lie bracket on P which is a bidifferential operator is the same as a Lie* bracket on the induced $\mathcal{D}$-module $P_{\mathcal{D}}$.

Examples. (i) The Lie bracket of vector fields defines a Lie* algebra structure on $\Theta_{\mathcal{D}} := \Theta_{X\mathcal{D}}$. For every $\mathcal{O}$-module F equipped with a Θ_X-action such that the action map is a differential operator with respect to Θ_X[70] the corresponding $\mathcal{D}$-module $F_{\mathcal{D}}$ is a $\Theta_{\mathcal{D}}$-module. E.g., $(\Omega_X^{\otimes j})_{\mathcal{D}}$ are $\Theta_{\mathcal{D}}$-modules.

(ii) If $\mathfrak{g}$ is a Lie algebra then $\mathfrak{g}_{\mathcal{O}} = \mathfrak{g} \otimes \mathcal{O}_X$ is a Lie $\mathcal{O}_X$-algebra, hence $\mathfrak{g}_{\mathcal{D}} := \mathfrak{g} \otimes \mathcal{D}_X = (\mathfrak{g}_{\mathcal{O}})_{\mathcal{D}}$ is a Lie* algebra.

(iii) If P is a coherent $\mathcal{O}$-module then then the sheaf of differential operators $\mathrm{Diff}(F,F)_X$ is an associative algebra in $\mathcal{D}iff(X)^*$ which acts on F. So $\mathrm{Diff}(F,F)_{\mathcal{D}} := \mathrm{Diff}(F,F)_{X\mathcal{D}}$ is an associative* (hence Lie*) algebra that acts on $F_{\mathcal{D}}$. This action yields a canonical isomorhism of associative* algebras $\mathrm{Diff}(F,F)_{\mathcal{D}} \xrightarrow{\sim} \mathcal{E}nd^*(F_{\mathcal{D}})$ (see 2.2.13, (2.2.11.1)).

Remark. One can show that every Lie* algebra can be represented X-locally as a quotient of a Lie* algebra which is free as a $\mathcal{D}$-module (hence, in particular, is an induced $\mathcal{D}$-module). For a proof see ???.

(c) Let R^ℓ be a finitely $\mathcal{D}_X$-presented $\mathcal{D}_X$-algebra.[71] Then $\Omega_{R/\mathcal{O}_X}$ is a finitely-presented $R^\ell[\mathcal{D}_X]$-module, so Θ_R is well-defined (see Remark (ii) in 2.3.12). This is a Lie* algebra (see 1.4.14). On the other hand, we have the Lie algebra algebra $E := \mathrm{Der}_{\mathcal{D}_X} R^\ell$ of horizontal $\mathcal{O}_X$-derivations of R^ℓ (experts in non-linear differential equations usually call E *the Lie-Bäcklund* algebra). We consider E as a sheaf on X. One has the canonical morphism of Lie algebras $h(\Theta_R) \to E$. If Ω_{R^ℓ} is a projective $R^\ell[\mathcal{D}_X]$-module (which happens if R^ℓ is $\mathcal{D}_X$-smooth) then this morphism is an isomorphism (see Remark (iii) in 2.3.12).

Θ_R is the first example of a Lie* R-algebroid (see 1.4.10 and 2.5.16 below).

[70] We assume that Θ_X acts on F as on an $\mathcal{O}$-module, so the action is automatically a differential operator with respect to F.

[71] Such R^ℓ essentially the same as a finite system of non-linear pde's of finitely many variables, see Remark in 2.3.1.

Remark. If Ω_R is a free $\mathcal{R}^\ell[\mathcal{D}_X]$-module then, choosing its generators $\nu_1, .., \nu_n$, we get an identification $E \xrightarrow{\sim} R^{\ell n}$, $\tau \mapsto (\langle \nu_i, \tau \rangle)$. It yields, in particular, an $\mathcal{O}_X$-module structure on E. With respect to this structure the bracket on E is a bidifferential operator, so we have the Lie* algebra $E_{\mathcal{D}}$. The action of E on R^ℓ is good (see 2.5.4), so we have the $E_{\mathcal{D}}$-action on R^ℓ. It yields an isomorphism of Lie* algebras $E_{\mathcal{D}} \xrightarrow{\sim} \Theta_R$.

(d) According to 2.2.14 for a coherent $\mathcal{D}$-module N a Lie cobracket $N \to N \otimes^! N$ yields a Lie* bracket on the dual $\mathcal{D}$-module N'. Assume that N is a $\mathcal{D}_X$-vector bundle; then this gives a 1-1 correspondence between the Lie$^!$ cobrackets on N (= Lie cobrackets on the left $\mathcal{D}$-module N^ℓ) and Lie* brackets on N'.

If N' is an induced $\mathcal{D}$-module, $N' = F_{\mathcal{D}}$ for a vector bundle F on X, then $N^\ell = {}_{\mathcal{D}}F^*$ (see 2.2.14), and our Lie cobracket is a skew-symmetric morphism ${}_{\mathcal{D}}F^* \to {}_{\mathcal{D}}F^* \otimes {}_{\mathcal{D}}F^*$, i.e., a section of $F \otimes \Lambda^2({}_{\mathcal{D}}F^*)$. Sometimes it is convenient to write Lie* brackets on $F_{\mathcal{D}}$ in this format.

Examples. (i) If G is a group $\mathcal{D}_X$-scheme such that its Lie coalgebra CoLie(G) is a coherent $\mathcal{D}$-module, then Lie(G) = CoLie(G)$'$ is a Lie* algebra. An action of G on a $\mathcal{D}_X$-scheme Y yields the action of Lie(G) on Y.

(ii) Consider the standard Lie* algebra structure on $\Theta_{\mathcal{D}}$ from (b)(i) above. The corresponding section of[72] $\Theta_X \otimes \Lambda^2({}_{\mathcal{D}}\Omega)$ is $\sum_{i,j} \partial_i \otimes ((1 \otimes dx^j) \wedge (\partial_j \otimes dx^i))$. Similarly, in the situation of (b)(ii) (we assume that $\dim \mathfrak{g} < \infty$) the corresponding section is the usual cobracket element in $\mathfrak{g}^* \otimes \Lambda^2 \mathfrak{g}^* \subset \mathfrak{g}^* \otimes \Lambda^2({}_{\mathcal{D}}\mathfrak{g}^*)$.

For a $\mathcal{D}$-module N set $h'(N) := \underline{\mathrm{Hom}}_{\mathcal{D}_X}(N, \omega_X) = \underline{\mathrm{Hom}}_{\mathcal{D}_X}(N^\ell, \mathcal{O}_X)$; this is a sheaf of k-vector spaces. The functor h' extends to a pseudo-tensor functor $h' : \mathcal{M}(X)^{!\circ} \to Sh(X)^{\otimes}$ in the obvious way (cf. (2.2.7.3)). The augmentation functor structure on h yields for $\mathcal{D}_X$-coherent N a canonical morphism $\alpha_N : h(N') \to h'(N)$ which extends to a morphism of pseudo-tensor functors in the obvious way. If N is a $\mathcal{D}_X$-vector bundle then α_N is an isomorphism.

[72] Here $\Omega = \Omega^1_X = \Theta^*_X$.

Therefore if N is a Lie$^!$ coalgebra then $h'(N)$ is a sheaf of Lie algebras; if N is $\mathcal{D}_X$-coherent then $h(N') \to h'(N)$ is a morphism of Lie algebras.

Example. Assume that our X is a symplectic variety with symplectic form ν. It yields a Poisson bracket on $\mathcal{O}_X$, so the corresponding induced $\mathcal{D}$-module $\mathcal{O}_\mathcal{D} = \mathcal{D}_X$ becomes a Lie* algebra; the Hamiltonian action morphism $\mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{\sim} \Theta_X$ (the latter identification is defined by ν) yields an embedding of Lie* algebras $a : \mathcal{O}_\mathcal{D} \to \Theta_\mathcal{D}$ whose image is the stabilizer of ν.[73] However the dual morphism of Lie coalgebras $a' : {}_\mathcal{D}\Omega \to {}_\mathcal{D}\mathcal{O} = \mathcal{D}_X$ is *not* surjective. Its image is a Lie coalgebra ${}_\mathcal{D}\Omega^\nu$ such that the image of embedding $h'({}_\mathcal{D}\Omega^\nu) \hookrightarrow h'({}_\mathcal{D}\Omega) = h(\Theta_\mathcal{D}) = \Theta_X$ is the Lie algebra of symplectic vector fields.

We see that the Lie algebra of symplectic vector fields comes from a Lie$^!$ coalgebra, not a Lie* algebra. So the world of Lie$^!$ coalgebras differs from that of Lie* algebras if we stop to consider $\mathcal{D}_X$-vector bundles (of course, we bring the unity back passing to derived categories). The examples below provide Lie* algebras which do not come from Lie$^!$ coalgebras.

In the next three examples (see 2.5.8–2.5.10*) we assume that* $\dim X = 1$. We will define canonical ω_X-extensions of some natural Lie* algebras of the type considered in 2.5.7(b).[74] The constructions follow the same pattern. Namely, to define our ω_X-extension $L^\flat$ of L we first construct an L-module extension $L^\natural$ of the adjoint representation by an L-module which equals $\omega_\mathcal{D} := \omega_X \otimes \mathcal{D}_X$ as a plain $\mathcal{D}$-module and has the property that the canonical morphism $\omega_\mathcal{D} \twoheadrightarrow \omega_X$ is L-invariant. Now $L^\flat$ is the push-out of $L^\natural$ by this arrow. The L-action on $L^\flat$ yields a $*$ bracket on $L^\flat$ which lifts the Lie* bracket on L. It satisfies automatically the Jacobi identity; the skew-symmetry follows from separate reasons.

Remark. For a Lie* algebra V any lisse $\mathcal{D}_X$-submodule of V lies in the center of V (see 2.2.4(ii)). Therefore any ω_X-extension of a Lie* algebra L is automatically central.

2.5.8 The Kac-Moody extension. Assume we are in situation of 2.5.7(b)(ii). Let $\kappa = (\,,\,)$ be an ad-invariant symmetric bilinear form on $\mathfrak{g}$. We will define an ω_X-extension $\mathfrak{g}^\kappa_\mathcal{D}$ of $\mathfrak{g}_\mathcal{D}$ called *the Kac-Moody* Lie* algebra.

[73] With respect to the action of $\Theta_\mathcal{D}$ on $\Omega^2_\mathcal{D}$, see Remark in 2.5.4.

[74] Notice that due to the presence of ω_X the corresponding $\mathcal{D}$-modules are *not* induced.

Consider the pairing $\tilde{\phi} : \mathfrak{g}_{\mathcal{O}} \times \mathfrak{g}_{\mathcal{O}} \to \omega_X$, $\tilde{\phi}(a,b) := (da,b)$ and the corresponding $*$ pairing $\tilde{\phi}_{\mathcal{D}} \in P_2^*(\{\mathfrak{g}_{\mathcal{D}}, \mathfrak{g}_{\mathcal{D}}\}, \omega_{\mathcal{D}})$; let $\phi \in P_2^*(\{\mathfrak{g}_{\mathcal{D}}, \mathfrak{g}_{\mathcal{D}}\}, \omega_X)$ be the composition of $\tilde{\phi}_{\mathcal{D}}$ with the canonical morphism $\omega_{\mathcal{D}} = \omega_X \otimes \mathcal{D}_X \to \omega_X$. Since κ is ad-invariant one has $\tilde{\phi}([a,b],c) = \tilde{\phi}(a,[b,c]) - \tilde{\phi}(b,[a,c])$. Since κ is symmetric one has $\tilde{\phi}(a,b) + \tilde{\phi}(b,a) = d(a,b)$, hence ϕ is skew-symmetric. Therefore ϕ is a 2-cocycle of the Lie* algebra $\mathfrak{g}_{\mathcal{D}}$. Our $\mathfrak{g}_{\mathcal{D}}^{\kappa}$ is the extension corresponding to this cocycle.

If $\mathfrak{g}$ is the Lie algebra of an algebraic group G and κ is Ad_G-invariant then the adjoint action of the sheaf of groups $G(\mathcal{O}_X)$ on $\mathfrak{g}_{\mathcal{D}}$ lifts canonically to $\mathfrak{g}_{\mathcal{D}}^{\kappa}$.[75] Namely, this action comes from an action of $G(\mathcal{O}_X)$ on $\mathfrak{g}_{\mathcal{O}} \oplus \omega_X$ defined by $g(a+\nu) := \mathrm{Ad}_g(a) + ((g^{-1}dg, a) + \nu)$. Therefore for every G-torsor $\mathfrak{F}$ on X we have the corresponding $\mathfrak{F}$-twisted Kac-Moody Lie* algebra which is an extension of the $\mathfrak{F}$-twisted Lie* algebra $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$ by ω_X. It can be described in "geometric" terms as follows.

Notice that the $\mathfrak{F}$-twist $\mathfrak{g}(\mathfrak{F})$ of $\mathfrak{g}_{\mathcal{O}}$ is the Lie algebra of infinitesimal symmetries of $\mathfrak{F}$, i.e., the Lie algebra of G-invariant "vertical" vector fields on $\mathfrak{F}$. Its dual $\mathfrak{g}(\mathfrak{F})^*$ is the sheaf of relative G-invariant 1-forms on $\mathfrak{F}/X$. Consider the sheaf $\mathcal{P}$ of G-invariant 1-forms on $\mathfrak{F}$. We have an extension of $\mathcal{O}_X$-modules $0 \to \omega_X \to \mathcal{P} \to \mathfrak{g}(\mathfrak{F})^* \to 0$. Let

$$0 \to \omega_X \to \mathcal{P}^{\kappa} \to \mathfrak{g}(\mathfrak{F}) \to 0 \tag{2.5.8.1}$$

be the pull-back of this extension by the morphism $\mathfrak{g}(\mathfrak{F}) \to \mathfrak{g}(\mathfrak{F})^*$ corresponding to κ.

Here is another way to define $\mathcal{P}^{\kappa}$. Consider the $\mathfrak{g}(\mathfrak{F}) \otimes \omega_X$-torsor $\mathcal{C}onn(\mathfrak{F})$ of connections on $\mathfrak{F}$; let $0 \to \mathfrak{g}(\mathfrak{F}) \otimes \omega_X \to \mathcal{C}onn(\mathfrak{F})\tilde{} \to \mathcal{O}_X \to 0$ be the corresponding $\mathcal{O}$-module extension. Tensoring by $\mathfrak{g}(\mathfrak{F})$ we get an extension of $\mathfrak{g}(\mathfrak{F})$ by $\mathfrak{g}(\mathfrak{F})^{\otimes 2} \otimes \omega_X$. Its push-out by $-\kappa \otimes id_{\omega} : \mathfrak{g}(\mathfrak{F})^{\otimes 2} \otimes \omega_X \to \omega_X$ equals $\mathcal{P}^{\kappa}$. The identification comes from the map $\mathcal{C}onn(\mathfrak{F})\tilde{} \otimes \mathfrak{g}(\mathfrak{F}) \to \mathcal{P}$ which sends $\nabla \otimes g$, $\nabla \in \mathcal{C}onn(\mathfrak{F})$, $g \in \mathfrak{g}(\mathfrak{F})$, to the 1-form on $\mathfrak{F}$ which vanishes on the ∇-horizontal vectors and whose restriction to fibers equals $\kappa(g) \in \mathfrak{g}(\mathfrak{F})^*$.

The Lie algebra $\mathfrak{g}(\mathfrak{F})$ acts on all the above $\mathcal{O}_X$-modules by $\mathcal{O}$-linear transformations. The action map is a differential operator of order ≤ 1 with respect to $\mathfrak{g}(\mathfrak{F})$. Therefore the

[75] More generally, for a Harish-Chandra pair $(\mathfrak{g}, K)$ and Ad-invariant κ the group $K(\mathcal{O}_X)$ acts on $\mathfrak{g}_{\mathcal{D}}^{\kappa}$.

Lie* algebra $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$ acts on $\mathcal{P}^{\kappa}_{\mathcal{D}}$. We define the extension of $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$-modules

$$0 \to \omega_X \to \mathfrak{g}(\mathfrak{F})^{\kappa}_{\mathcal{D}} \xrightarrow{\pi} \mathfrak{g}(\mathfrak{F})_{\mathcal{D}} \to 0 \tag{2.5.8.2}$$

as the push-out of the extension $\mathcal{P}^{\kappa}_{\mathcal{D}}$ by the canonical morphism of $\mathcal{D}$-modules $\omega_X \otimes \mathcal{D} \to \omega_X$. The $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$-action yields a binary $*$ operation $[\ \ ]$ on $\mathfrak{g}(\mathfrak{F})^{\kappa}_{\mathcal{D}}$, $[a,b] := \pi(a)b$.

Every trivialization of $\mathfrak{F}$ yields an identification $\mathfrak{g}(\mathfrak{F}) \xrightarrow{\sim} \mathfrak{g}_{\mathcal{O}}$ and a splitting $\mathcal{P} \xrightarrow{\sim} \mathfrak{g}(\mathfrak{F})^* \oplus \omega_X$, hence an isomorphism $\mathfrak{g}(\mathfrak{F})^{\kappa}_{\mathcal{D}} \xrightarrow{\sim} \mathfrak{g}_{\mathcal{D}} \oplus \omega_X = \mathfrak{g}^{\kappa}_{\mathcal{D}}$. We leave it to the reader to check that it identifies $[\ \ ]$ with the Lie* bracket on $\mathfrak{g}^{\kappa}_{\mathcal{D}}$ (in particular, we see that $[\ \ ]$ is a Lie* bracket) and the action of $\mathrm{Aut}\mathfrak{F}$ on $\mathfrak{g}(\mathfrak{F})^{\kappa}_{\mathcal{D}}$ with the above $G(\mathcal{O}_X)$-action on $\mathfrak{g}^{\kappa}_{\mathcal{D}}$.[76] Thus $\mathfrak{g}(\mathfrak{F})^{\kappa}_{\mathcal{D}}$ is the above $\mathfrak{F}$-twisted Kac-Moody Lie* algebra.

Remark. $\mathfrak{g}(\mathfrak{F})$ is the $\mathcal{O}_X$-linear part of the Lie $\mathcal{O}_X$-algebroid $\mathcal{A}(\mathfrak{F})$ of all infinitesimal symmetries of $(\mathfrak{F}, X)$. Our $\mathfrak{g}(\mathfrak{F})^{\kappa}_{\mathcal{D}}$ comes from a canonical ω_X-extension of $\mathcal{A}(\mathfrak{F})_{\mathcal{D}}$. For every connection ∇ on $\mathfrak{F}$ this extension canonically splits over $\nabla(\Theta_{\mathcal{D}})$. See [BS] where a more general setting is discussed.

2.5.9 The trace extension. Let F be a vector bundle on X. Denote by $\mathcal{A}(F)$ the Lie algebroid of infinitesimal symmetries of (F, X) which is an extension of Θ_X by the Lie algebra $\mathfrak{g}l(F)$ of endomorphisms of F. So $\mathcal{A}(F)$ is a Lie subalgebra of the algebra $\mathrm{Diff}(F,F)_X$ of differential operators which consists of operators of first order whose principal symbol lies in $id_F \cdot \Theta_X \subset (\mathrm{End}F) \otimes \Theta_X$.

We will define a canonical ω_X-extension $\mathcal{A}(F)^{\flat}_{\mathcal{D}}$ of the Lie* algebra $\mathcal{A}(F)_{\mathcal{D}}$. The construction goes as follows (see [BS] for details).

Set $F^\circ := F^* \otimes \omega_X$. The adjunction of differential operators yields a canonical antiisomorphism $\mathrm{Diff}(F,F)_X \xrightarrow{\sim} \mathrm{Diff}(F^\circ, F^\circ)_X$, so F° is a right $\mathrm{Diff}(F,F)_X$-module. Recall that one has a canonical identification of $\mathrm{Diff}(F,F)_X$-bimodules

$$\mathrm{Diff}(F,F)_X \xrightarrow{\sim} (j_*j^*F \boxtimes F^\circ)/F \boxtimes F^\circ \tag{2.5.9.1}$$

which maps $1 \in \mathrm{Diff}(F,F)$ to $1 \in \mathrm{End}F = F \boxtimes F^\circ(\Delta)/F \boxtimes F^\circ$; the inverse map assigns to a "kernel" $k(x,y) \in j_*j^*F \boxtimes F^\circ$ the differential operator $f \mapsto k(f), k(f)(x) :=$

[76] $\mathrm{Aut}\mathfrak{F}$ is identified with $G(\mathcal{O}_X)$ by the trivialization of $\mathfrak{F}$.

$\mathrm{Res}^{(y)}_{y=x}\langle k(x,y), f(y)\rangle$. Therefore $(j_*j^*F\boxtimes F^\circ)/F\boxtimes F^\circ(-\Delta)$ is an extension of $\mathrm{Diff}(F,F)_X$ by $F\otimes F^\circ$. Pushing it out by the trace map $tr : F\otimes F^\circ \to \omega_X$ we get an ω_X-extension; restricting it to $\mathcal{A}(F)\subset \mathrm{Diff}(F,F)_X$ we get an extension

$$0\to\omega_X\to\mathcal{A}(F)^\natural\xrightarrow{\pi^\natural}\mathcal{A}(F)\to 0. \tag{2.5.9.2}$$

The above constructions were natural, so $\mathcal{A}(F)$ acts on all the above objects. Thus (2.5.9.2) is an extension of $\mathcal{A}(F)$-modules (where $\mathcal{A}(F)$ acts on ω_X via the standard Θ_X-action). By construction, $\mathcal{A}(F)^\natural$ is an $\mathcal{O}_{X\times X}$-module supported on the diagonal, so, by 2.1.8(i), it is an object of the category $\mathcal{D}iff(X)$ (see 2.1.7). The $\mathcal{A}(F)$-action on $\mathcal{A}(F)^\natural$ is given by a bidifferential operator (see 2.2.4), so we have the $\mathcal{A}(F)_\mathcal{D}$-module extension $\mathcal{A}(F)^\natural_\mathcal{D}$ of the adjoint representation of $\mathcal{A}(F)_\mathcal{D}$ by $\omega_\mathcal{D}$. Its push-out by the canonical morphism $\omega_\mathcal{D}\twoheadrightarrow\omega_X$ is our extension

$$0\to\omega_X\to\mathcal{A}(F)^\flat\xrightarrow{\pi}\mathcal{A}(F)_\mathcal{D}\to 0. \tag{2.5.9.3}$$

The $\mathcal{A}(F)_\mathcal{D}$-module structure on $\mathcal{A}(F)^\flat_\mathcal{D}$ yields a $*$ operation $[\ \ ]$ on it, $[a,b]:=\pi(a)b$. It is easy to see that $[\ \ ]$ is skew-symmetric,[77] so it is a Lie* bracket.

Remarks. (i) The trace extension satisfies the following additivity property. Namely, let $\mathcal{F}$ be a flag $F_1\subset ..\subset F_n=F$ such that gr_iF are vector bundles. Let $\mathcal{A}(F)_\mathcal{F}\subset\mathcal{A}(F)$ be the subalgebroid of symmetries preserving $\mathcal{F}$. Let $\mathcal{A}(F)^\flat_{\mathcal{F}\mathcal{D}}$ be the restriction of the extension $\mathcal{A}(F)^\flat_\mathcal{D}$, and let $\mathcal{A}(F)^{\flat'}_{\mathcal{F}\mathcal{D}}$ be the Baer sum of the pull-backs of the extensions $\mathcal{A}(\mathrm{gr}_iF)^\flat_\mathcal{D}$ by the obvious projections $\mathcal{A}(F)_\mathcal{F}\to\mathcal{A}(\mathrm{gr}_iF)$. now there is an obvious canonical isomorphism of ω_X-extensions

$$\mathcal{A}(F)^\flat_{\mathcal{F}\mathcal{D}}\xrightarrow{\sim}\mathcal{A}(F)^\flat_{\mathcal{F}\mathcal{D}}. \tag{2.5.9.4}$$

(ii) The canonical identification of Lie algebroids $\mathcal{A}(F)\xrightarrow{\sim}\mathcal{A}(F^\circ)$ lifts to a canonical identification of the ω_X-extensions

$$\mathcal{A}(F)^\flat_\mathcal{D}\xrightarrow{\sim}\mathcal{A}(F^\circ)^\flat_\mathcal{D} \tag{2.5.9.5}$$

[77] Indeed, for every $\alpha,\beta\in\mathcal{A}(F)^\natural$ one has $\pi^\natural(\alpha)\beta+\pi^\natural(\beta)\alpha\in\omega_X\subset\mathcal{A}(F)^\natural$ and this 1-form is exact, see [BS] ??.

induced by the transposition of coordinates symmetry of $X \times X$.

(iii) We have two natural ω_X-extensions of the Lie* algebra $\mathfrak{g}l(F)_{\mathcal{D}}$: the first one is the Kac-Moody extension $\mathfrak{g}l(F)^{\kappa}_{\mathcal{D}}$ where κ is form $a, b \mapsto -\mathrm{tr}(ab)$ on $\mathfrak{g}l$, the second one is the restriction $\mathfrak{g}l(F)^{\flat}_{\mathcal{D}}$ of $\mathcal{A}(F)^{\flat}_{\mathcal{D}}$ to $\mathfrak{g}l(F)_{\mathcal{D}} \subset \mathcal{A}(F)_{\mathcal{D}}$. Now there is a canonical isomorphism of ω_X-extensions

$$\mathfrak{g}l(F\omega_X^{\otimes 1/2})^{\flat}_{\mathcal{D}} \xrightarrow{\sim} \mathfrak{g}l(F)^{\kappa}_{\mathcal{D}} \tag{2.5.9.6}$$

which lifts the obvious identification $\mathfrak{g}l(F\omega_X^{\otimes 1/2}) \xrightarrow{\sim} \mathfrak{g}l(F)$. Namely, every trivialization of F yields the $\mathcal{D}$-module splittings of both extensions,[78] and (2.5.9.6) is compatible with these trivializations.

(iv) Assume that F is a Θ_X-equivariant bundle such that the Θ_X-action morphism $\Theta_X \to \mathcal{A}(F)$ is a differential operator.[79] Then we can restrict $\mathcal{A}(F)^{\flat}_{\mathcal{D}}$ to $\Theta_{\mathcal{D}} \subset \mathcal{A}(F)_{\mathcal{D}}$. Thus every such F yields an ω_X-extension of $\Theta_{\mathcal{D}}$ which is a Lie* subalgebra of $\mathcal{A}(F)^{\flat}_{\mathcal{D}}$.

2.5.10 The Virasoro extension. We will define a canonical ω_X-extension $\Theta^{v}_{\mathcal{D}}$ of the Lie* algebra $\Theta_{\mathcal{D}}$ called *the Virasoro* Lie* algebra. The construction is similar to that of the Kac-Moody extension with the torsor of connections on $\mathfrak{F}$ replaced by the torsor of projective connections on X. The definition of projective connection is recalled in (b) below after a technical digression of (a); the Virasoro extension is constructed in (c); it is identified in (d) with an extension from Remark (iii) of 2.5.9.

(a) Let $X^{(n)}$ the nth infinitesimal neighbourhood of the diagonal $X \hookrightarrow X \times X$ for some $n \geq 0$, so $\mathcal{O}_{X^{(n)}} := \mathcal{O}_{X\times X}/\mathcal{I}^{n+1}$ where $\mathcal{I}$ is the ideal of the diagonal. The scheme $X^{(n)}$ carries a canonical involution σ (transposition of coordinates); the action of the Lie algebra Θ_X on $X = X^{(0)}$ extends in the obvious (diagonal) way to an action on $X^{(n)}$ commuting with σ. An $\mathcal{O}$-module E on $X^{(n)}$ yields, by 2.1.8(i), the $\mathcal{D}$-module $E_{\mathcal{D}}$ on X. If E is σ-equivariant then σ acts on $E_{\mathcal{D}}$. We have the plain sheaf of σ-invariants E^{σ} and the $\mathcal{D}$-module one $E^{\sigma}_{\mathcal{D}} := (E_{\mathcal{D}})^{\sigma}$.

[78] The splitting of the Kac-Moody extension was discussed in 2.5.8. The splitting of the trace extension in case $\mathrm{rk}F = 1$ identifies $\mathfrak{g}l(\omega_X^{\otimes 1/2})_{\mathcal{D}}$ with the antiinvariants of the involution (2.5.9.5) of $\mathcal{A}(\omega_X^{\otimes 1/2})^{\flat}_{\mathcal{D}}$. Use (2.5.9.4) to reduce the general situation to the rank 1 case.

[79] i.e., the Θ_X-action satisfies the condition from 2.5.7(b)(i).

The projection $\mathcal{O}^*_X \leftarrow \mathcal{O}^*_{X^{(n)}}$ has a canonical section $\mathfrak{s} : \mathcal{O}^*_X \to \mathcal{O}^*_{X^{(n)}}$ uniquely defined by properties that it has étale local nature and $\mathfrak{s}(f^2) = f \boxtimes f|_{X^{(n)}}$. It is clear that $\mathfrak{s}$ is σ-invariant and commutes with the action of Θ_X. Explicit formula: $\mathfrak{s}(f)(x,y) = f(x) + \frac{1}{2}f'(x)(y-x) + \frac{1}{4}(f''(x) - f'(x)^2/f(x))(y-x)^2 + ...$ So a line bundle $\mathcal{L}$ on X yields a σ-equivariant line bundle $\mathcal{L}^{\mathfrak{s}}$ on $X^{(n)}$. To construct $\mathcal{L}^{\mathfrak{s}}$ one should choose (locally) a square root $\mathcal{L}^{\otimes 1/2}$; then $\mathcal{L}^{\mathfrak{s}} = \mathcal{L}^{\otimes 1/2} \boxtimes \mathcal{L}^{\otimes 1/2}|_{X^{(n)}}$. Notice that the canonical filtration $\mathcal{J}^i\mathcal{L}^{\mathfrak{s}}$ has successive quotients equal to $\mathcal{L}$, $\mathcal{L} \otimes \omega_X$,.., $\mathcal{L} \otimes \omega_X^{\otimes n}$, and σ acts on ith one as multiplication by $(-1)^i$. If Θ_X acts on $\mathcal{L}$ then $\mathcal{L}^{\mathfrak{s}}$ is a (σ, Θ_X)-equivariant line bundle.

Denote by ν the restriction of $\mathcal{O}_{X\times X}(\Delta)$ on $X^{(n)}$. This is a (σ, Θ_X)-equivariant line bundle on $X^{(n)}$: the action of σ is the transposition action multiplied by -1, and Θ_X acts in the obvious way. One has $\nu^{\otimes m} = \mathcal{O}_{X\times X}(m\Delta)/\mathcal{O}_{X\times X}((m-n-1)\Delta)$.

(b) From now on $n = 2$. Assume that we have a (σ, Θ_X)-equivariant line bundle λ on $X^{(2)}$ whose restriction to X equals $\mathcal{O}_X$ (with the trivial action of σ and the standard action of Θ_X). The canonical filtration on λ has successive quotients $\mathcal{O}_X$, ω_X, $\omega_X^{\otimes 2}$, and σ acts on them as multiplication by, respectively, $1, -1, 1$. Let $\mathfrak{P}(\lambda) \subset \lambda^\sigma$ be the preimage of $1 \in \mathcal{O}_X$ by the projection $\lambda^\sigma \to \mathcal{O}_X$. This is a Θ_X-equivariant $\omega_X^{\otimes 2}$-torsor. It is clear that $\mathfrak{P}(\lambda^{\otimes m}) = m\mathfrak{P}(\lambda)$.

Our key example is $\lambda = \omega_X^{\mathfrak{s}} \otimes \nu$. The elements of $\mathfrak{P} := \mathfrak{P}(\omega_X^{\mathfrak{s}} \otimes \nu)$ are called *projective connections* on X.

Remarks. (i) A local coordinate t on X yields a projective connection ρ_t defined as restriction of $(x-y)^{-1}dx^{1/2}dy^{1/2}$ to $X^{(2)}$.[80] An immediate calculation shows that $(f(t)\partial_t)(\rho_t) = \frac{1}{12}f'''(x)dx^2$.

(ii) We have $(\omega_X^{\mathfrak{s}} \otimes \nu)^{\otimes m} = (\omega_X^{\otimes m/2} \boxtimes \omega_X^{\otimes m/2})(m\Delta)/((m-3)\Delta)$. So for $m = 3, 4$ one has embeddings $m\mathfrak{P} \hookrightarrow (j_*j^*\omega_X^{\otimes m/2} \boxtimes \omega_X^{\otimes m/2})/\omega_X^{\otimes m/2} \boxtimes \omega_X^{\otimes m/2} = \mathrm{Diff}(\omega_X^{\otimes 1-m/2}, \omega_X^{\otimes m/2})$ which identify projective connections[81] with symmetric differential operators $\omega_X^{\otimes -1/2} \to \omega_X^{\otimes 3/2}$ of order 2 and principal symbol 1 and, respectively, skew-symmetric differential operators $\omega_X^{\otimes -1} \to \omega_X^{\otimes 2}$ of order 3 and principal symbol 1. For an interpretation of projective

[80] Here x, y are coordinates on $X \times X$ corresponding to t.

[81] Here we identify $\mathfrak{P}$ with $3\mathfrak{P}, 4\mathfrak{P}$ by means of multiplications by 3 and 4.

connections as $\mathfrak{sl}_2$-opers see [BD].

(c) Tensoring Θ_X by $\mathfrak{P}$ we get an $\mathcal{O}$-module extension $\mathcal{P}^v$ of Θ_X by $\Theta_X \otimes \omega_X^{\otimes 2} = \omega_X$. It carries a canonical action of Θ_X which coincides with the standard action on the subquotients ω_X, Θ_X. According to Remark (i), every local coordinate t yields a splitting of our extension; in terms of this splitting the Θ_X-action is given by the cocycle $f(t)\partial_t \mapsto \frac{1}{12}f'''(t)dt^2$. Now the Virasoro extension

$$0 \to \omega_X \to \Theta^v_{\mathcal{D}} \xrightarrow{\pi} \Theta_{\mathcal{D}} \to 0 \tag{2.5.10.1}$$

is the push-out of the extension $\mathcal{P}^v_{\mathcal{D}}$ by the canonical morphism of $\mathcal{D}$-modules $\omega_{\mathcal{D}} \to \omega_X$.

The action of Θ_X on $\mathcal{P}^v$ is given by a bidifferential operator, so it yields a $*$ action of the Lie* algebra $\Theta_{\mathcal{D}}$ on $\mathcal{P}^v_{\mathcal{D}}$ and $\Theta^v_{\mathcal{D}}$ which lifts the adjoint action. The binary $*$ operation $[\ \ ]$, $[a,b] := \pi(a)b$, on $\Theta^v_{\mathcal{D}}$ is a Lie* bracket. Indeed, it obviously satisfies the Jacobi identity; one proves the skew-symmetry using the above explicit formula.

(d) In Remark (iv) of 2.5.9 consider $F = \mathcal{O}_X$. We get an ω_X-extension of $\Theta_{\mathcal{D}}$ which identifies canonically with the twice Virasoro extension[82] $\Theta^{2v}_{\mathcal{D}}$. Namely, the embedding $\Theta^{2v}_{\mathcal{D}} \hookrightarrow \mathcal{A}(\mathcal{O}_X)^{\flat}_{\mathcal{D}}$ comes from the morphism $\mathcal{P}^v \to \mathcal{A}(\mathcal{O}_X)^{\natural}$, $\tau \otimes \rho \mapsto (\tau \otimes 1)\rho^2$.[83]

Remark. More generally, the ω_X-extension of $\Theta_{\mathcal{D}}$ which comes from the Θ_X-equivariant line bundle $\omega_X^{\otimes j}$ identifies canonically with $\Theta^{2(6j^2-6j+1)v}_{\mathcal{D}}$. See e.g. [BS]?? for explicit "coordinate" formulas for this identification.

2.5.11. Let L be a Lie* algebra, $x \in X$ a (closed) point, $j_x : U_x := X \smallsetminus \{x\} \hookrightarrow X$. The vector space $h(L)_x$ carries the Ξ_x-topology (see 2.1.13).

Lemma. (i) The Lie bracket on $h(L)_x$ is Ξ_x-continuous with respect to each variable.

(ii) If j_x^*L is a countably generated $\mathcal{D}_{U_x}$-module then the bracket is Ξ_x-continuous.

Proof. See 2.2.17(i),(ii). □

Remark. If we decide to play with better objects $\tilde{h}_x$ from Remark (iii) of 2.1.14 then the countability condition becomes irrelevant.

[82] i.e., the push-out of the Virasoro extension by the multiplication by 2 map $\omega_X \to \omega_X$.

[83] Here $\rho^2 \in \omega_X^{\boxtimes 2}(2\Delta)/\omega_X^{\boxtimes 2}(-\Delta)$.

Examples. If we are in situation of 2.5.7(b) and the restriction of P to the complement of x is $\mathcal{O}$-coherent then $\hat{h}_x(P_{\mathcal{D}}) = P \otimes O_x$ (see 2.1.14(ii)) and the Lie bracket comes from the bidifferential operator defining the Lie bracket on P.

In the situation of 2.5.8 the Lie algebra $\hat{h}(j_{x*}j_x^*\mathfrak{g}_{\mathcal{D}}^{\kappa})$ is the central extension of $\mathfrak{g}(K_x) = \hat{h}(j_{x*}j_x^*\mathfrak{g}_{\mathcal{D}})$ by $k = \hat{h}(j_{x*}j_x^*\omega_X)$ defined by 2-cocycle $a, b \mapsto \mathrm{Res}_x(da, b)$, i.e., the usual affine Kac-Moody algebra.

2.5.12. Denote by $\Xi_x^{Lie}(L)$ the topology on L at x (see 2.1.13 for terminology) whose base consists of those $L_\xi \in \Xi_x(L)$ which are Lie* subalgebras of L. As follows from the Remark in 2.2.7, the latter condition amounts to the fact that $h(L_\xi)_x$ is a Lie subalgebra of $h(L)_x$. Therefore one can describe the Ξ_x^{Lie}-topology on $h(L)_x$ as the topology whose base is formed by all Ξ_x-open Lie subalgebras of $h(L)_x$.

The Lie bracket on $h(L)_x$ is automatically continuous with respect to Ξ_x^{Lie}-topology. Indeed, the base of our topology is formed by Lie subalgebras, and for every open Lie subalgebra V the adjoint action of V on $h(L)_x/V$ is continuous by 2.5.11(i).

Therefore the completion $\hat{h}_x^{Lie}(L)$ is a topological Lie algebra, and $\hat{h}_x(L)$ is a topological Lie algebra if L satisfies the condition of 2.5.11(ii). We have a canonical continuous morphism $\hat{h}_x(L) \to \hat{h}_x^{Lie}(L)$ of Lie algebras.

The two topologies do not differ if L satisfies a coherency condition:

2.5.13 Lemma. (i) Let M be an L-module such that its restriction to U_x is a coherent $\mathcal{D}$-module. Then for any $M_\xi, M_{\xi'} \in \Xi_x(M)$ such that $M_{\xi'}$ is a coherent $\mathcal{D}_X$-module there exists $L_\zeta \in \Xi_x^{Lie}(L)$ such that $L_\zeta(M_{\xi'}) \subset M_\xi$.

(ii) If the restriction of L to U_x is a coherent $\mathcal{D}$-module then the topologies $\Xi_x(L)$ and $\Xi_x^{Lie}(L)$ coincide.

Remark. Using the terminology of 2.6.9 the above statements can be reformulated as follows. Recall that $F := \hat{h}_x(M)$ is a Tate vector space (see 2.1.13 and 2.6.9 for terminology), and (i) just says that the action morphism $h(L)_x \to \mathrm{End}\hat{h}_x(M)$ is continuous with respect to the Ξ_x^{Lie}-topology and the topology on $\mathrm{End}F$ introduced in 2.6.9. In the situation of (ii) $\hat{h}_x(L)$ is a topological Lie algebra which is a Tate vector space. The

argument below actually proves that any such object has a base of topology formed by open Lie subalgebras.

Proof. (i) Replacing M_ξ by $M_\xi \cap M_{\xi'}$ we may assume that $M_\xi \subset M_{\xi'}$. According to 2.2.17(iii) the maximal sub $\mathcal{D}$-module $L_\zeta \subset L$ such that $L_\zeta(M_{\xi'}) \subset M_\xi$ belongs to $\Xi_x(L)$. It is also a Lie* algebra, so we are done.

(ii) We want to show that every $L_\xi \in \Xi_x(L)$ contains some $L_\zeta \in \Xi_x^{Lie}(L)$. We can assume that L_ξ is a coherent $\mathcal{D}$-module. According to the proof of (i) the normalizer N of L_ξ is open. Set $L_\zeta := N \cap L_\xi$. □

In the non-coherent situation the topologies Ξ_x and Ξ_x^{Lie} can be very different:

2.5.14 Example. Consider the Lie* algebra $L = \mathfrak{g}l(\mathcal{D}_X)$. Then $h(L)$ is $\mathcal{D}_X$ considered as a sheaf of Lie algebras. For every $x \in X$ the completion $\hat{h}_x(L)$ is a Lie algebra because L satisfies the condition of 2.5.11(ii). Our L is an induced $\mathcal{D}_X$-module, so, by Example (i) of 2.1.13, the completion $\hat{h}_x(L)$ is the Lie algebra $D = \mathcal{D}_{O_x}$ of differential operators with coefficients in O_x (the formal completion of the local ring at x). Denote by $D^{\leq n}$ the subspace of differential operators of order $\leq n$; this is a free O_x-module of rank $n+1$. A vector subspace of D is open in Ξ_x-topology if its intersection with every $D^{\leq n}$ is open. The Ξ_x^{Lie}-topology is defined by Ξ_x-open Lie subalgebras. It is easy to see that for every open subspace $P \subset D^{\leq 2}$ the Lie subalgebra generated by P is open. Such subalgebras form a base of the topology Ξ_x^{Lie}; in particular, the topology Ξ_x^{Lie}, as opposed to Ξ_x, has a countable base.

Let us describe the Ξ_x^{Lie}-topology on D explicitly assuming that $\dim X = 1$. Let t be a local parameter at x, so $O_x = k[[t]]$. Set $\partial = \partial_t$. For $a \geq 0$, $b \geq 1$ let $D^{a,b} \subset D$ be the space of differential operators $\Sigma \phi_i(t)\partial^i$, $\phi_i(t) \in t^{a+bi}O_x$.

Lemma. $D^{a,b}$ form a basis of the Ξ_x^{Lie}-topology on D.

Proof. Indeed, $D^{a,b}$ are open in the Ξ_x-topology and they are associative, hence Lie, subalgebras of D. It remains to check that any open Lie subalgebra $Q \subset D$ contains some $D^{a,b}$. Set $D_i^{a,b} := D^{a,b} \cap D^{\leq i}$. A simple calculation shows that the operator

$A_b := ad_{t^{b+1}\partial^2} \in \mathrm{End}D$ preserves $D^{a,b}$. It sends $D^{\leq i}$ to $D^{\leq i+1}$ and the corresponding operator $\mathrm{gr}_i D^{a,b} \to \mathrm{gr}_{i+1} D^{a,b}$ is an isomorphism if $a \geq 1, i \geq 0$. Therefore $D^{a,b} = \underset{i \geq o}{\oplus} A_b^i(t^a O_x)$. Now choose $a, b \geq 1$ such that $t^a O_x \subset Q$ and $t^{b+1}\partial^2 \in Q$. Then $D^{a,b} \subset Q$. □

We see that the completion $\hat{h}_x^{Lie}(L)$ consists of "differential operators of infinite order" $\Sigma \phi_i(t)\partial^i$ such that the number of the first non-vanishing coefficient of $\phi_i(t)$ grows faster than any linear function of i.

Remark. We will see in 2.6.13 that the above L carries an important topology which is weaker then Ξ_x^{Lie}.

2.5.15. Let L be any Lie* algebra. Notice that for an $h(L)_x$-module V the action map $h(L)_x \times V \to V$ is continuous with respect to the Ξ_x^{Lie}-topology (and the discrete topology on V) if and and only if it is continuous with respect to the Ξ_x-topology (since the stabilizer of every $v \in V$ is a Lie subalgebra).[84] Such $h(L)_x$-modules are called *discrete*; the corresponding category is denoted by $\hat{h}_x(L)$mod.

Let $\mathcal{M}(X, L)_x$ be the category of L-modules supported at x. For $M \in \mathcal{M}(X, L)_x$ the Lie algebra $h(L)_x$ acts canonically on M, hence $i_x^! M = h(M)$ is an $h(L)_x$-module. By 2.5.13(i) it is discrete.

Lemma. The functor $h : \mathcal{M}(X, L)_x \to \hat{h}_x(L)$mod is an equivalence of categories.

Proof. Follows from 2.5.4. □

2.5.16 Lie* algebroids. Let R^ℓ be a commutative $\mathcal{D}_X$-algebra. According to 1.4.10 we have the notion of Lie* R-algebroid. As follows from the second lemma of 2.3.12 these objects have étale local nature. Therefore we know what is a Lie* algebroid $\mathcal{L}$ on an algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ and what are $\mathcal{L}$-modules. Then $h(\mathcal{L})$ is a sheaf of Lie algebras (on $\mathcal{Y}_{\acute{e}t}$) acting on $\mathcal{O}_\mathcal{Y}$ by horizontal $\mathcal{O}_X$-derivations.

The constructions of 1.4.13 are étale local as well (for the same reason). So for a Lie* algebroid $\mathcal{L}$ on $\mathcal{Y}$ which is a $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$ (see 2.3.11) we have a commutative DG superalgebra $\mathcal{C}(\mathcal{L})_\mathcal{Y}$ called *the de Rham-Chevalley complex of* $\mathcal{L}$, and for an $\mathcal{L}$-module M

[84] In fact, one can carachterize Ξ_x^{Lie} as the weakest topology on $h(L)_x$ which satisfies this property.

we have a DG $\mathcal{C}(\mathcal{L})_{\mathcal{Y}}$-module $\mathcal{C}(\mathcal{L},M)_{\mathcal{Y}}$. If we forget about the differential then $\mathcal{C}(\mathcal{L})_{\mathcal{Y}} = \mathrm{Sym}(\mathcal{L}'[-1])$, $\mathcal{C}(\mathcal{L},M)_{\mathcal{Y}} = \mathrm{Sym}(\mathcal{L}'[-1])\otimes M$ where $\mathcal{L}'$ is the $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$ dual to $\mathcal{L}$.

For example, if $\Omega_{\mathcal{Y}}$ is finitely-presented then the tangent Lie algebroid $\Theta_{\mathcal{Y}}$ is well-defined (see 1.4.14 and Remarks in 2.3.12). If $\Omega_{\mathcal{Y}}$ is a $\mathcal{D}_{\mathcal{Y}}$-vector bundle, e.g. if $\mathcal{Y}$ is smooth (see 2.3.13–2.3.16), then $h(\Theta_{\mathcal{Y}})$ is the Lie algebra of all horizontal $\mathcal{O}_X$-derivations of $\mathcal{O}_{\mathcal{Y}}$. One has $\mathcal{C}(\Theta_{\mathcal{Y}})_{\mathcal{Y}} = DR_{\mathcal{Y}/X}$ (the relative de Rham complex).

More generally, if V is a $\mathcal{D}_X$-vector bundle on such $\mathcal{Y}$ then the Lie* algebroid $\mathcal{E}(V)$ on $\mathcal{Y}$ (see 1.4.14) is well-defined. The morphism $\tau\colon \mathcal{E}(V)\to\Theta_{\mathcal{Y}}$ is surjective, so $\mathcal{E}(V)$ is an extension of $\Theta_{\mathcal{Y}}$ by $\mathfrak{g}l(V)$.

The material of 2.5.11–2.5.13 generalizes to the setting of Lie* algebroids as follows. We assume that $\dim X = 1$, R^ℓ is a commutative $\mathcal{D}$-algebra on U_x, $\mathcal{L}$ a Lie* R-algebroid. Being an R^ℓ-module, $j_{x*}\mathcal{L}$ carries the topology Ξ_x^R at x (see Remark in 2.4.8).

2.5.17 Lemma. (i) The action of $h_x(j_{x*}\mathcal{L})$ on $h_x(j_{x*}R)$ is Ξ_x^R -Ξ_x^{as}-continuous with respect to each variable. The Lie bracket on $h_x(j_{x*}\mathcal{L})_x$ is Ξ_x^R-continuous with respect to each variable.

(ii) If R^ℓ is a countably generated $\mathcal{D}_{U_x}$-algebra and $\mathcal{L}$ is a countably generated $R^\ell[\mathcal{D}_{U_x}]$-module then the action is Ξ_x^R -Ξ_x^{as}-continuous and the bracket is Ξ_x^R-continuous.

Proof. (i) Take $R_\xi\in\Xi_x^{as}(j_{x*}R)$. For $\bar l\in h_x(j_{x*}\mathcal{L})$ one has $\{r\in R_\xi : \bar l^{-1}(r)\in R_\xi\}\in\Xi_x^{as}$. For $\bar r\in h_x(R)$ one has $\{l\in\mathcal{L} : \bar l(\bar r)\in h_x(R_\xi)\}\in\Xi_x^R$.

(ii) Take R_ξ as above and $\mathcal{L}_\zeta\in\Xi_x^R$ which is an R_ξ-module. It suffices to find $\mathcal{L}_\eta\in\Xi_x^R$, $R_\nu\in\Xi_x^{as}$ such that the action map on $\mathcal{L}_\eta$, R_ν takes values in R_ξ, and the bracket on $\mathcal{L}_\eta$ takes values in $\mathcal{L}_\zeta$. Together with (i) this proves the continuity.

Let $P\subset R_\xi$, $Q\subset\mathcal{L}_\zeta$ be countably generated $\mathcal{D}$-submodules such that $P|_{U_x}$ generates R^ℓ as a $\mathcal{D}_{U_x}$-algebra and $Q|_{U_x}$ generates $\mathcal{L}$ as an $R^\ell[\mathcal{D}_{U_x}]$-module. According to 2.2.17(ii) there exists $P'\in\Xi_x(P)$, $Q'\in\Xi_x(Q)$ such that the action map on Q',P' takes values in R_ξ, and the bracket on Q' takes values in $\mathcal{L}_\zeta$. Now let R_ν be the $\mathcal{D}_X$-subalgebra of $j_{x*}R$ generated by P', $\mathcal{L}_\eta$ be the R_ν-submodule of $j_{x*}\mathcal{L}$ generated by Q'. □

We see that countability conditions of 2.5.17(ii) imply that $\hat{h}_x^R(\mathcal{L})$ is a topological Lie R_x^{as}-algebroid.

Remark. If we decide to play with better objects $\tilde{h}_x$ from Remark (iii) of 2.1.14 then the countability condition becomes irrelevant.

The module $j_{x*}\mathcal{L}$ also carries a weaker topology $\Xi_x^{R\,Lie} := \Xi_x^R \cap \Xi_x^{Lie}$.

2.5.18 Lemma. Assume that R^ℓ is finitely generated $\mathcal{D}_{U_x}$-algebra and $\mathcal{L}$ is a finitely generated $R^\ell[\mathcal{D}_{U_x}]$-module. Then the Ξ_x^R- and $\Xi_x^{R\,Lie}$-topologies on $h_x(\mathcal{L})$-coincide.

Proof. Similar to that of 2.5.13; left to the reader. □

§6 The Tate extension

In this section it is assumed that dim $X = 1$. We are going to define a canonical central extension of a matrix Lie* algebra. A linear algebra version of this ubiquitous extension first appeared (implicitly) in Tate's remarkable note [T]; it was rediscovered and studied under various disguises and names[85] in the beginning of 80s (see [DJKM], [KP], and [SW]).

We begin in 2.6.1 with a general format for construction of central extensions of Lie algebras. The Tate extension in $\mathcal{D}$-module setting is defined in 2.6.2, 2.6.3. Tate's original linear algebra construction is presented in 2.6.9–2.6.11; the two constructions are related in 2.6.12–2.6.15.

We will see in 3.8.6, 3.8.18 that the Tate extension arises naturally as a part of (chiral) Clifford algebra.

2.6.1 Some concrete nonsense. We live in an abelian pseudo-tensor k-category. For simplicity of notation we write formulas in terms of "symbolic elements" of our objects.

(i) Let L be a Lie algebra. Denote by $\mathcal{E}(L)$ the category of pairs $(L^\natural, \pi)$ where $L^\natural$ is an L-module, $\pi : L^\natural \to L$ a surjective morphism of L-modules (we consider L as an L-module with respect to the adjoint action). We often write $L^\natural$ for $(L^\natural, \pi)$ and set $L_0^\natural := \mathrm{Ker}\pi$.

[85] For example, in Moscow they used to call it "the Japanese extension"; cf. A. Brehm's discourse in " The Life of Animals" on the names of the common cockroach (Blatta germanica) in different tongues: Russians call the fellow Prussian, in Austrian highlands it is known as Russian, etc.

An object $L^\natural \in \mathcal{E}(L)$ yields a symmetric L-invariant pairing $(\ \ ) \in P_2(\{L^\natural, L^\natural\}, L_0^\natural)$, $(a,b) := \pi(a)b + \pi(b)a$. Notice that the L-action on $L_0^\natural$ is trivial if and only if $(\ \ ) \in P_2(\{L,L\}, L_0^\natural) \subset P_2(\{L^\natural, L^\natural\}, L_0^\natural)$. We say that $L^\natural$ is *central* if $(\ \ ) = 0$.

Denote by $\mathcal{CE}(L)$ the category of central Lie algebra extensions of L. Notice that for every $L^\flat \in \mathcal{CE}(L)$ the adjoint action of $L^\flat$ factors through L. So $L^\flat$ is an L-module; the structure projection $L^\flat \to L$ makes it an object of $\mathcal{E}(L)$.

Lemma. The functor $\mathcal{CE}(L) \to \mathcal{E}(L)$ we have defined is fully faithful. It identifies $\mathcal{CE}(L)$ with the category of central objects in $\mathcal{E}(L)$. □

Assume we have $L^\natural \in \mathcal{E}(L)$ and an L-invariant morphism $\mathrm{tr} : L_0^\natural \to F$ (the L-action on F is trivial). We say that tr is *central* if $\mathrm{tr}(\ \ ) \in P_2(\{L,L\}, F)$ equals 0. Denote by $L^\natural_{\mathrm{tr}}$ the push-forward of $L^\natural$ by tr; this is an object of $\mathcal{E}(L)$ which is an extension of L by F. It is clear that $L^\natural_{\mathrm{tr}} \in \mathcal{CE}(L)$, if and only if tr is central.

Remark. If every L-invariant symmetric F-valued pairing on L is trivial then every tr as above is automatically central.

We suggest the reader to skip the rest of this subsection returning to it when necessary.

(ii) Here is an example of extensions that arise in the above manner:

Lemma. Let L be a Lie algebra, $L_\pm \subset L$ ideals such that $L_+ + L_- = L$, and $\mathrm{tr} : L_0 := L_+ \cap L_- \to F$ a morphism such that $\mathrm{tr}([l_+, l_-]) = 0$ for every $l_\pm \in L_\pm$. Then there exists a central extension $L^\flat$ of L by F together with sections $s_\pm : L_\pm \to L^\flat$ such that the images $s_\pm(L_\pm)$ are ideals in $L^\flat$[86] and for any $l \in L_0$ one has $(s_+ - s_-)l = \mathrm{tr}(l) \in F \subset L^\flat$. Such $(L^\flat, s_\pm)$ is unique (up to a unique isomorphism).

Proof. The uniqueness of $(L^\flat, s_\pm)$ is clear. Let us construct $L^\flat$. Set $L^\natural := L_+ \oplus L_-$, and consider the extension of L-modules $0 \to L_0 \xrightarrow{\alpha} L^\natural \xrightarrow{\beta} L \to 0$ where $\alpha(l) = (l, -l)$, $\beta(l_+, l_-) = l_+ + l_-$ (the action of L is the adjoint one). Then $L^\natural \in \mathcal{E}(L)$ and tr is a central morphism (the L-invariance of tr and vanishing of $\mathrm{tr}(\ \ )$ follow from the property $\mathrm{tr}([L_+, L_-]) = 0$). Now set $L^\flat := L^\natural_{\mathrm{tr}}$. □

Remark. In the situation of the corollary one has $L_+/L_0 \xrightarrow{\sim} L/L_-$. The Lie algebra

[86] Or, equivalently, that $s_\pm$ are morphisms of L-modules (with respect to the adjoint actions).

L_+/L_0 has a central extension $(L_+/L_0)^\flat$ defined as the push-out of the extension $0 \to L_0 \to L_+ \to L_+/L_0 \to 0$ by $\mathrm{tr} : L_0 \to F$. Now s_+ yields an isomorphism $(L_+/L_0)^\flat \xrightarrow{\sim} L^\flat/s_-(L_-)$.

Similarly, s_- yields an isomorphism $(L_-/L_0)^\flat \xrightarrow{\sim} L^\flat/s_+(L_+)$ where the central extension $(L_-/L_0)^\flat$ of L_-/L_0 is the push-out of the extension $0 \to L_0 \to L_- \to L_-/L_0 \to 0$ by $-\mathrm{tr} : L_0 \to F$.

(iii) Let A be an associative algebra. Denote by A^{Lie} our A considered as a Lie algebra.

If $A^\flat$ is a central extension of A^{Lie} by F then its bracket comes from a skew-symetric pairing $[\ \ ]^\flat \in P_2(\{A,A\}, A^\flat$. We say that $A^\flat$ satisfies *the cyclic property* if for every $a,b,c \in A$ one has $[ab,c]^\flat + [bc,a]^\flat + [ca,b]^\flat = 0$.

Denote by $\mathcal{E}(A)$ the category of pairs $(A^\natural, \pi)$ where $A^\natural$ is an A-bimodule, $\pi : A^\natural \to A$ a surjective morphism of A-bimodules. Notice that any A-bimodule is an A^{Lie}-module with respect to the commutator action. Thus every object of $\mathcal{E}(A)$ can be considered as an object of $\mathcal{E}(A^{Lie})$, i.e., we have a faithful functor $\mathcal{E}(A) \to \mathcal{E}(A^{Lie})$.

Take $A^\natural \in \mathcal{E}(A)$ and let $\mathrm{tr} : A_0^\natural \to F$ be a central A^{Lie}-invariant morphism, so $A^\flat := A^\natural_{\mathrm{tr}}$ is a central Lie algebra extension of A^{Lie} by F.

Lemma. $A^\flat$ satisfies the cyclic property.

Proof. Let $\tilde{c}$ be a lifting of c. We have a tautological identity $[ab,\tilde{c}] + [b\tilde{c},a] + [\tilde{c}a,b] = 0$ in $A^\natural$ (it holds in any A-bimodule).[87] Now notice that $\mathrm{tr}([bc,\tilde{a}] - [b\tilde{c},a]) = \mathrm{tr}(a,bc) = 0$ (since $b\tilde{c}$ is a lifting of bc) and similarly $\mathrm{tr}([ca,\tilde{b}] - [\tilde{c}a,b]) = \mathrm{tr}(ca,b) = 0$. □

For example, if we have two-sided ideals $A_\pm \subset A$ and a morphism $\mathrm{tr} : A_0 := A_+ \cap A_- \to F$ which vanishes on $[A_+, A_-] \subset A_0$ then the central extension $A^\flat$ of A^{Lie} defined in (ii) above satisfies the cyclic property.

Remarks. (i) Assume that our pseudo-tensor category is a tensor unital category (say, the category of k-modules) and A is a commutative unital algebra. Consider the A-

[87] This identity can be interpreted as follows. For any A-bimodule M and $a \in A$ denote by l_a, r_a the left, resp. right, multiplication by a; set $ad_a := l_a - r_a$. The identity says that $ad_{ab} = ad_a l_b + ad_b r_a$. Define the A-bimodule structure on $\mathrm{End}(M)$ by $af := fr_a$, $fb := fl_b$. Then the identity says that $ad_{ab} = ad_a b + a ad_b$, i.e., the map $A \to \mathrm{End}(M)$, $a \mapsto ad_a$, is a derivation.

module of differentials Ω_A, so we have a universal derivation $d : A \to \Omega_A$. For every object F the map $\mathrm{Hom}(\Omega_A, F) \to P_2(\{A, A\}, F)$, which sends $\varphi : \Omega_A \to F$ to the operation $a, b \mapsto \varphi(adb)$ is injective, and its image consists of those operations ψ that $\psi(a, bc) = \psi(ab, c) + \psi(ac, b)$. Notice that φ vanishes on the image of d if and only if the corresponding ψ is skew-symmetric. Now for $(A^\natural, \mathrm{tr})$ as above one has $[\quad]^\flat \in P_2(\{A, A\}, F)$ (since A is commutative), and our lemma shows that $[\quad]^\flat$ yields a morphism $\Omega_A/d(A) \to F$. This is one of the principal ideas of Tate's work [T].

(ii) ?? TSYGAN'S PICTURE -DOPISAT'

(iv) Assume our pseudo-tensor category is augmented. For A, $A^\natural$ as above $h(A)$ is an associative algebra and $A^\natural$ is an $h(A)$-bimodule (see 1.2.8). The A-actions also yield morphisms $A \otimes h(A^\natural) \to A^\natural$, $h(A^\natural) \otimes A \to A^\natural$ which are morphisms of, respectively, $(A, h(A))$- and $(h(A), A)$-bimodules.

If $z^\natural \in h(A)$ is such that $z := \pi(z^\natural) \in h(A)$ is an $A^\natural$-central element (i.e., left and right multiplications by z on $A^\natural$ coincide) then the morphism $\partial_{z^\natural} : A \to A_0^\natural$, $a \mapsto az^\natural - z^\natural a$, is a derivation. If $\mathrm{tr} : A_0^\natural \to F$ is an A^{Lie}-invariant morphism then the composition $\mathrm{tr}\partial_{z^\natural} : A \to F$ depends only on z, so we denote it by $\mathrm{tr}\partial_z$. It is clear that $\mathrm{tr}\partial_z$ kills the commutant $[A, A]$.

Let us assume that A is a unital associative algera (see 1.2.8), $A^\natural$ is a unital bimodule, and $1 = 1_A \in h(A)$ lifts to $h(A^\natural)$. We get a canonical morphism $\mathrm{tr}\partial_1 : A \to F$.

Lemma. One has $\mathrm{tr}(a, b) = \mathrm{tr}\partial_1(ab)$. Therefore tr is central if and only if $\mathrm{tr}\partial_1 = 0$.

Proof. Choose $\tilde{1} \in h(A^\natural)$. We have two sections $A \to A^\natural$, $a \mapsto 1^\natural a$, $a1^\natural$. One has $(1^\natural a, b1^\natural) = ab1^\natural - 1^\natural ab$. Applying tr we get the lemma. □

In particular, if A^{Lie} is perfect, i.e., $[A, A] = A$, then every tr is central.

2.6.2 The Tate extension in $\mathcal{D}$-module setting. Let V, V' be $\mathcal{D}$-modules on X, $\langle\quad\rangle \in P_2^*(\{V, V'\}, \omega_X) = P_2^*(\{V', V\}, \omega_X)$ a $*$ pairing. By 1.4.2 the $\mathcal{D}$-module $\mathcal{G} := V \otimes V'$ is an associative* algebra which acts on V, V'. We have the corresponding Lie* algebra $\mathcal{G}^{Lie}$; the pairing is $\mathcal{G}^{Lie}$-invariant.

The Tate extension is a canonical central extension of Lie* algebras

$$0 \to \omega_X \to \mathcal{G}^\flat \xrightarrow{\pi^\flat} \mathcal{G}^{Lie} \to 0. \tag{2.6.2.1}$$

To define $\mathcal{G}^\flat$ consider the exact sequence[88] of $\mathcal{D}$-modules on $X \times X$

$$V \boxtimes V' \xrightarrow{\varepsilon} j_* j^*(V \boxtimes V') \xrightarrow{\pi} \Delta_*(V \otimes V') \to 0. \tag{2.6.2.2}$$

Here $\Delta: X \hookrightarrow X \times X$ is the diagonal embedding, $j: U := X \times X \smallsetminus \Delta(X) \hookrightarrow X \times X$ is the complementary embedding, and π is the canonical arrow. Namely, according to (2.2.2.1), one has $V \otimes V' = H^1 \Delta^!(V \boxtimes V') = \operatorname{Coker} \varepsilon$; explicitly, π sends $(t_2 - t_1)^{-1} v \boxtimes v' \in j_* j^* V \boxtimes V'$ to $v \otimes v'(dt)^{-1} \in \Delta.(V \otimes V') \subset \Delta_*(V \otimes V')$. Note that $\langle \quad \rangle$ vanishes on $\operatorname{Ker} \varepsilon$.[89] So, pushing out the above exact sequence by $\langle \quad \rangle$, we get an extension of $\Delta_*(V \otimes V')$ by $\Delta_* \omega_X$. This extension is supported on the diagonal; applying $\Delta^!$ we get the Tate extension $\mathcal{G}^\flat$. We denote the canonical morphism $j_* j^*(V \boxtimes V') \to \Delta_* \mathcal{G}^\flat$ by $\mu = \mu_{\mathcal{G}}$.

Let us define now the Lie* bracket $[\quad]_{\mathcal{G}^\flat}$ on $\mathcal{G}^\flat$.

Let L be a Lie* algebra that acts on V and V' preserving the pairing. Then the sheaf of Lie algebras $h(L)$ acts on $\mathcal{G}$. This action lifts canonically to an action on $\mathcal{G}^\flat$. Indeed, global sections of $h(L)$ act on V, V' preserving the $*$ pairing, hence they act on $j_* j^*(V \boxtimes V')$ and $\mathcal{G}^\flat$ by functoriality, and everything is compatible with localization of X. In particular, the canonical action of $\mathcal{G}$ on V, V' yields a canonical action of $h(\mathcal{G}^{Lie})$ on $\mathcal{G}^\flat$.

2.6.3 Proposition. There is a unique Lie* algebra structure on $\mathcal{G}^\flat$ such that the adjoint action of $h(\mathcal{G}^\flat)$ on $\mathcal{G}^\flat$ is the above action composed with $h(\mathcal{G}^\flat) \to h(\mathcal{G}^{Lie})$. The projection $\mathcal{G}^\flat \to \mathcal{G}^{Lie}$ is a morphism of Lie* algebras, so $\mathcal{G}^\flat$ is a central extension of $\mathcal{G}^{Lie}$. This extension satisfies the cyclic property (see 2.6.1(iii)).

Proof. The uniqueness is clear, as well as the latter property (notice that the commutator on $\mathcal{G}^\flat$ comes from the action of $\mathcal{G}^{Lie}$ on $\mathcal{G}^\flat$ that lifts the adjoint action).

Let us show that our picture fits into the format of 2.6.1(iii); this implies the existence statement. The cyclic property follows then from the lemma in 2.6.1(iii).

[88] It is also exact from the left if V, V' have no $\mathcal{O}_X$-torsion.

[89] Indeed, $\operatorname{Ker} \varepsilon = \Delta_* H^{-1}(V \overset{L}{\otimes} V') = \Delta_* \operatorname{Tor}_1^{\mathcal{O}_X}(V, V' \omega_X^{-1})$, hence its sections have finite supports.

Consider the ordered set of all submodules $N_\alpha \subset j_*j^*V \boxtimes V'$ such that $j_*j^*V \boxtimes V'/N_\alpha$ is supported on the diagonal. Set $\mathcal{G}^\natural_\alpha := \Delta^!(j_*j^*V \boxtimes V'/N_\alpha)$ and $\mathcal{G}^\natural := \varprojlim \mathcal{G}^\natural_\alpha$.

The pro$\mathcal{D}_X$-module $\mathcal{G}^\natural$ carries a canonical structure of topological $\mathcal{G}$-bimodule. This means that for every *coherent* $\mathcal{D}_X$-submodule $\mathcal{G}_\beta \subset \mathcal{G}$ we have mutually commuting $*$ pairings (left and right action) in $\cdot^l \in P_2^*(\{\mathcal{G}_\beta, \mathcal{G}^\natural\}, \mathcal{G}^\natural)$ and $\cdot^r \in P_2^*(\{\mathcal{G}^\natural, \mathcal{G}_\beta\}, \mathcal{G}^\natural)$ which are compatible with respect to embeddings of $\mathcal{G}_\beta$'s and satisfy the left and right $\mathcal{G}$-action properties.[90]

To define $\cdot^l$ consider the morphism $\cdot_V \boxtimes id_{V'} : \mathcal{G} \boxtimes V \boxtimes V' \to \Delta_*^{1=2}(V \boxtimes V')$. Here $\cdot_V$ is the $*$ action of $\mathcal{G}$ on V and $\Delta^{1=2}$ is the diagonal embedding $(x, y) \mapsto (x, x, y)$. Localizing we get the morphism $\tilde{\cdot}^l : \mathcal{G} \boxtimes j_*j^*(V \boxtimes V') \to \Delta_*^{1=2} j_*j^*(V \boxtimes V')$. It satisfies the following continuity property: for every N_α, $\mathcal{G}_\beta$ as above there exists $N_{\alpha'}$ such that $\tilde{\cdot}^l(\mathcal{G}_\beta \boxtimes N_\alpha) \subset \Delta_*^{1=2}(N_\alpha)$.[91] Now $\cdot^l$ is the completion of $\tilde{\cdot}^l$ with respect to the N_α-topology.

One defines $\cdot^r$ in the similar way using the action of $\mathcal{G}$ on V' instead of V. It is clear that $\cdot^{l,r}$ define on $\mathcal{G}^\natural$ a topological $\mathcal{G}$-bimodule structure.

The projection π from (2.6.2.2) yields the projection $\mathcal{G}^\natural \twoheadrightarrow \mathcal{G}$ which we also denote by π. This is a morphism of (topological) $\mathcal{G}$-bimodules. The $*$ pairing $\langle \quad \rangle$ yields a $\mathcal{G}^{Lie}$-invariant morphism $\mathcal{G}^\natural_0 := \operatorname{Ker} \pi \to \omega_X$ which we denote by tr.

One has[92] $\mathcal{G}^\flat = \mathcal{G}^\natural_{\mathrm{tr}}$, and the $h(\mathcal{G}^{Lie})$-module structure on $\mathcal{G}^\flat$ defined in 2.6.2 comes from the $\mathcal{G}^\flat$-module structure on $\mathcal{G}^\natural_{\mathrm{tr}}$. Therefore, by 2.6.1(i), our proposition follows from the next Lemma:

Lemma. tr is a central morphism.

Proof of Lemma. (a) It suffices to check our Lemma in case $V = \mathcal{D}_X^n$, $V' = \omega\mathcal{D}_X^n$ is its dual. Indeed, our problem is local, so we can choose a surjective morphism $\mathcal{D}_X^n \twoheadrightarrow V$. Then $\langle \quad \rangle$ yields a morphism $V' \to \omega\mathcal{D}_X^n$. Our constructions are functorial with respect

[90] This makes sense since for every two coherent $\mathcal{D}_X$-submodules $\mathcal{G}_\beta$, $\mathcal{G}_{\beta'}$ of $\mathcal{G}$ their product (i.e., the image in $\mathcal{G}$ of $\mathcal{G}_\beta \boxtimes \mathcal{G}_{\beta'}$ by the product morphism) is coherent.

[91] Let g_i be a finite set of generators of $\mathcal{G}_\beta$, so we have the correponding classes $\bar{g}_i \in h(\mathcal{G}_\beta)$ and endomorphisms $\bar{g}_i \tilde{\cdot}^l$ of $j_*j^*V \boxtimes V'$. Our $N_{\alpha'}$ is the intersection of preimages of N_α by these endomorphisms.

[92] We use notation of 2.6.1(i); the material of 2.6.1 renders to the setting of topological modules in the obvious way.

to morphisms of $(V, V', \langle\ \ \rangle)$'s. Looking at $(V, V') \leftarrow (\mathcal{D}_X^n, V') \to (\mathcal{D}_X^n, \omega\mathcal{D}_X^n)$ we get the desired reduction.

(b) For such V, V' our $\mathcal{G}$ is the unital associative algebra and $\mathcal{G}^\natural$ is a unital $\mathcal{G}$-module. One has $\mathcal{G}^{Lie} = \mathfrak{g}l(\mathcal{D}_X^n)$ (see 2.5.7(a)), so, by 2.6.1(iv), it remains to check the following fact:

Sublemma. The Lie* algebra $\mathfrak{g}l(\mathcal{D}_X^n)$ is perfect.

Proof of Sublemma. It suffices to consider the case $n = 1$. One has $\mathfrak{g}l(\mathcal{D}_X) = D_{\mathcal{D}}^{Lie}$ where D^{Lie} is the sheaf $D = \mathcal{D}_X$ of differential operators considered as a Lie algebra in $\mathcal{D}iff(X)^*$, see 2.5.7(b)(iii). We want to show that the adjoint action of $D^{Lie} = h(D_{\mathcal{D}}^{Lie})$ on $D_{\mathcal{D}}$ has trivial coinvariants. In fact, the action of the subalgebra $\Theta_X \subset D^{Lie}$ of vector fields has trivial coinvariants. To see this notice that this action preserves the filtration by the degree of the differential operator, and $\mathrm{gr}D_{\mathcal{D}} = \underset{n\geq 0}{\oplus} \Theta_{X\mathcal{D}}^{\otimes n}$. Let us show that the coinvariants of the Θ_X action on $\Theta_{\mathcal{D}}^{\otimes n}$ are trivial for $n \neq -1$. Since we deal with a coherent $\mathcal{D}$-module (see Lemma(i) in 2.1.6) this amounts to the equality $[\Theta_X, \Theta_X^{\otimes n}] = \Theta_X^{\otimes n}$. If $v \in \Theta_X$, $\eta \in \omega_X$ then $[v, v^{n+1}\eta] = v^{n+1}[v, \eta] = v^{n+1}d(v, \eta)$. Such sections for $v(x) \neq 0$ generate the stalk of $\Theta_X^{\otimes n}$ at x if $n \neq -1$. □

2.6.4. If V is a $\mathcal{D}_X$-vector bundle (see 2.2.13), V' its dual, and $\langle\ \ \rangle$ is the canonical pairing, then $\mathcal{G}^{Lie} = \mathfrak{g}l(V)$ (see 2.5.7(a)). Therefore we have a canonical central extension $\mathfrak{g}l(V)^\flat$ of $\mathfrak{g}l(V)$ by ω_X. Passing to homology, we get a central extension $\bar{\mathfrak{g}}l(V)^\flat$ of the sheaf of Lie algebras $\bar{\mathfrak{g}}l(V) = h(\mathfrak{g}l(V))$ (which is the sheaf of endomorphisms of V considered as a Lie algebra) by $h(\omega_X)$.

The Tate extension satisfies an important additivity property. Namely, let $\mathcal{F}$ be a finite flag $V_1 \subset .. \subset V_n = V$ such that every $\mathrm{gr}_i V$ is a $\mathcal{D}_X$-vector bundle. Let $\mathfrak{g}l(V)_{\mathcal{F}} \subset \mathfrak{g}l(V)$ be the corresponding parabolic Lie* subalgebra, so we have the projections $\mathfrak{g}l(V)_{\mathcal{F}} \to \mathfrak{g}l(\mathrm{gr}_i V)$. Let $\mathfrak{g}l(V)_{\mathcal{F}}^\flat$ be the restriction of the Tate extension $\mathfrak{g}l(V)^\flat$ to $\mathfrak{g}l(V)_{\mathcal{F}}$, and let $\mathfrak{g}l(V)_{\mathcal{F}}^{\flat'}$ be the sum of the pull-backs of the Tate extensions $\mathfrak{g}l(\mathrm{gr}_i V)^\flat$. Now there is a canonical isomorphism of extensions

$$\mathfrak{g}l(V)_{\mathcal{F}}^\flat \xrightarrow{\sim} \mathfrak{g}l(V)_{\mathcal{F}}^{\flat'}. \tag{2.6.4.1}$$

To see this consider the submodule $P := \Sigma V_j \boxtimes (V/V_{j-1})' \subset V \boxtimes V'$; we have the obvious projections $\pi_i : P \to \mathrm{gr}_i V \boxtimes (\mathrm{gr}_i V)'$. Now $\mathfrak{g}l(V)_{\mathcal{F}} = \Sigma V_i \otimes (V/V_{i-1})' \subset V \otimes V'$, i.e., $\mathfrak{g}l(V)_{\mathcal{F}} = \Delta^! P \subset \Delta^!(V \boxtimes V') = V \otimes V'$. So the exact sequence $0 \to \Delta_* \omega_X \to \mathfrak{g}l(V)^\flat_{\mathcal{F}} \to \mathfrak{g}l(V)_{\mathcal{F}} \to 0$ is the push-out of $0 \to P \to j_* j^* P \to \Delta^! P \to 0$ by $f : P \to \Delta_* \omega_X$, where f is the restriction of the canonical pairing $V \boxtimes V' \to \Delta_* \omega_X$. Since f is the sum of the canonical pairings $\mathrm{gr}_i V \boxtimes (\mathrm{gr}_i V)' \to \Delta_* \omega_X$ composed with π_i we get (2.6.4.1). We leave it to the reader to check compatibility with Lie* brackets.

2.6.5. Assume that V is induced, so $V = F_{\mathcal{D}} := F \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$, F is a locally free $\mathcal{O}_X$-module of finite rank. We have $\mathfrak{g}l(V) = \mathrm{Diff}(F,F)_{\mathcal{D}}$ and $\bar{\mathfrak{g}}l(V) = \mathrm{Diff}(F,F)_X$ (see 2.5.7(b)(iii)). Let us the $h(\omega_X)$-extension $\bar{\mathfrak{g}}l(V)^\flat$. The dual $\mathcal{D}$-module V' equals $F^\circ_{\mathcal{D}}$ where $F^\circ = F^* \omega_X$, so $j_* j^*(V \boxtimes V') = (j_* j^* F \boxtimes F^\circ)_{\mathcal{D}}$ and $\Delta_*(V \otimes V') = (j_* j^*(F \boxtimes F^\circ)/F \boxtimes F^\circ)_{\mathcal{D}}$.

Applying h, we see that $\Delta^{\cdot} \bar{\mathfrak{g}}l(V)^\flat$ is equal to the push-forward of the extension $0 \to F \boxtimes F^\circ \to j_* j^* F \boxtimes F^\circ \to j_* j^* F \boxtimes F^\circ / F \boxtimes F^\circ \to 0$ by the map $F \boxtimes F^\circ \to \Delta^{\cdot} \omega_X$, $f(x) \boxtimes g(y) \mapsto \langle f(x), g(x) \rangle$. The identification $\Delta^{\cdot} j_* j^*(F \boxtimes F^o)/F \boxtimes F^o \xrightarrow{\sim} \mathrm{Diff}(F,F)_X$ is the isomorphism (2.5.9.1). The left and right action of $\bar{\mathfrak{g}}l(V) = \mathrm{Diff}(F,F)_X$ on $\Delta^{\cdot} h(j_* j^*(V \boxtimes V')) = \Delta^{\cdot} j_* j^*(F \boxtimes F^\circ)$ come from the usual left and right action of $\mathrm{Diff}(F,F)_X$ on F and F^o, respectively; the adjoint action yields our Lie bracket on the quotient $\bar{\mathfrak{g}}l(V)^\flat$. Therefore $\bar{\mathfrak{g}}l(V)^\flat$ coincides with the canonical extension of the Lie algebra $\mathrm{Diff}(F,F)_X$ as defined in, say, [BS] ??.

Notice that, by construction, the embedding $\mathcal{A}(F) \hookrightarrow \mathrm{Diff}(F,F)_X$ lifts to the one of ω_X-extensions $\mathcal{A}(F)^\flat_{\mathcal{D}} \hookrightarrow \mathfrak{g}l(V)^\flat$. In particular, by 2.5.10 (d), $\mathfrak{g}l(\mathcal{D}_X)$ contains the Virasoro extension $\Theta^{2v}_{\mathcal{D}}$.

2.6.6. The construction of Tate's extension renders easily to the case of A^ℓ-modules, $A^\ell \in \mathcal{C}omu^!(X)$. Assume we have $V, V' \in \mathcal{M}(X, A^\ell)$ and $\langle \quad \rangle \in P^*_{A2}(\{V, V'\}, A)$. We have the associative* matrix algebra[93] $V \underset{A}{\otimes} V' \in \mathcal{M}(X, A)$; denote by $\mathcal{G}_A$ the corresponding Lie* algebra. We will define a canonical central extension of Lie* algebras in $\mathcal{M}(X, A)$

$$0 \to A \to \mathcal{G}^\flat_A \to \mathcal{G}_A \to 0. \tag{2.6.6.1}$$

[93] As usual, we write the tensor product of right $A[\mathcal{D}_X]$-modules as $\underset{A}{\otimes}$ instead of $\underset{A}{\otimes}^!$.

Let us construct $\mathcal{G}_A^\flat$ as a plain $A^\ell[\mathcal{D}_X]$-module.

To construct $\mathcal{G}_A^\flat$ we first define an extension of $\mathcal{D}$-modules $0 \to A \to \mathcal{G}_A^{'\flat} \to V \otimes^! V' \to 0$. Namely, $\Delta_* \mathcal{G}_A^{'\flat}$ is push-forward of $V \boxtimes V' \to j_* j^* V \boxtimes V' \to \Delta_*(V \otimes^! V') \to 0$ by $\langle \quad \rangle : V \boxtimes V' \to \Delta_* A$. This $\mathcal{G}_A^{'\flat}$ is an $A \otimes^! A$-module; set $\mathcal{G}_A^\flat = \mathcal{G}_A^{'\flat} / I \mathcal{G}_A^{'\flat}$ where $I := \mathrm{Ker}(A \otimes^! A \to A)$. One defines the bracket on $\mathcal{G}_A^\flat$ as in 2.6.2, 2.6.3; again, we refer to 3.8 for a natural Clifford realization of $\mathcal{G}_A^\flat$.

The above construction is compatible with étale localization, so it works in the setting of algebraic $\mathcal{D}_X$-spaces. Namely, let $\mathcal{Y}$ be a algebraic $\mathcal{D}_X$-space, V, V' be $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules equipped with an $\mathcal{O}_\mathcal{Y}$-valued $*$ pairing $\langle \quad \rangle$; assume that $\mathcal{Y}$ is flat over X and $\mathcal{T}or_{\mathcal{O}_\mathcal{Y}}(V, V') = 0$. We have a Lie* algebra $\mathcal{G}_\mathcal{Y} = V \otimes^! V' \in \mathcal{M}_(\mathcal{Y})$ and its Tate (central) extension $\mathcal{G}_\mathcal{Y}^\flat$ by $\mathcal{O}_\mathcal{Y}$. Notice that the construction of $\mathcal{G}_\mathcal{Y}$, $\mathcal{G}_\mathcal{Y}^\flat$ is compatible with base change (with respect to morphisms of $\mathcal{Y}$'s).

Assume now that $\mathcal{Y}$ is a smooth algebraic $\mathcal{D}_X$-space, V is a $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$, V' is its $*$ dual, so $\mathcal{G}_\mathcal{Y} = \mathfrak{g}l(V)$. The algebroid $\mathcal{E}(V)$ (see 2.3.16) acts on V, V'. As in 2.6.2 its action on $\mathfrak{g}l(V)$ lifts canonically to an action on $\mathfrak{g}l(V)^\flat$. On $\mathfrak{g}l(V) \subset \mathcal{E}(V)$ this action coincides with the "adjoint" one.

Remark. One may see that $\mathfrak{g}l(V)$ coincides with its commutant, so every central extension of $\mathfrak{g}l(V)$ has no non-trivial automorphisms.

The meaning of the following definition will become clear in 3.8.18.

2.6.7 Definition. A *Tate structure* on V is the following datum (i), (ii):

(i) An extension $\mathcal{E}(V)^\flat$ of the Lie* algebroid $\mathcal{E}(V)$ on $\mathcal{Y}$ by $\mathcal{O}_\mathcal{Y}$.

The corresponding Lie* $\mathcal{O}_\mathcal{Y}$-algebra $\mathcal{E}(V)^{\flat\circ} = \mathrm{Ker}(\tau)$ is an extension of $\mathfrak{g}l(V)$ by $\mathcal{O}_\mathcal{Y}$.

(ii) An identification of this extension with $\mathfrak{g}l(V)^\flat$ compatible with $\mathcal{E}(V)^\flat$ actions.

Here $\mathcal{E}(V)^\flat$ acts on $\mathcal{E}(V)^{\flat\circ}$ via the adjoint action, and on $\mathfrak{g}l(V)^\flat$ via the canonical action of $\mathcal{E}(V)$.

Tate structures form a groupoid $\mathcal{T}ate(V) = \mathcal{T}ate(\mathcal{Y}, V)$. They have étale local nature, so we have a sheaf of groupoids $\mathcal{T}ate(V)_{\mathcal{Y}_{\text{ét}}}$ on $\mathcal{Y}_{\text{ét}}$.

For a Lie* algebroid L on $\mathcal{Y}$ we denote by $\mathcal{P}^{cl}(L) = \mathcal{P}^{cl}(\mathcal{Y}, L)$[94] the groupoid of Lie* algebroid extensions of L by $\mathcal{O}_{\mathcal{Y}}$. This isa Picard groupoid, or rather a k-vector space in categories, in the obvious way (the sum is the Baer sum operation). The objects of $\mathcal{P}^{cl}(L)$ have étale local nature, so we have a sheaf $\mathcal{P}^{cl}(L)_{\mathcal{Y}_{\text{ét}}}$ of Picard groupoids on $\mathcal{Y}_{\text{ét}}$.

There is a canonical Action of $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$ on $\mathcal{T}ate(V)$. Namely, an object Θ^{ν} of $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$ amounts, by pull-back, to an extension $\mathcal{E}(V)^{\nu}$ of $\mathcal{E}(V)$ by $\mathcal{O}_{\mathcal{Y}}$ trivialized over $\mathfrak{g}l(V)$. The corresponding automorphism of $\mathcal{T}ate(V)$ sends $\mathcal{E}(V)^{\flat}$ to its Baer sum with $\mathcal{E}(V)^{\nu}$.

2.6.8 Lemma. $\mathcal{T}ate(V)$ is a $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$-Torsor. □

In the rest of this section we present a version of the original Tate's linear algebra construction and relate it then with the above Lie* algebra one.

2.6.9 Tate's linear algebra. Let F be a k-vector space equipped with a separated complete linear[95] topology. A closed vector subspace $L \subset F$ is *compact*, resp. *cocompact*, if for any open vector subspace $U \subset F$ one has $\dim L/(L \cap U) < \infty$, resp. $\dim F/(L+U) < \infty$. A *c-lattice* is an open compact subspace; dually, a *d-lattice* is a discrete cocompact subspace.

It is easy to see that F has a c-lattice if and only if it has a d-lattice. Such F is called *Tate vector space*.[96] For a Tate vector space F its dual[97] F^* is again a Tate space; the canonical map $F \to F^{**}$ is an isomorphism.

Any discrete vector space[98] is a Tate space. Duality interchanges discrete and compact Tate spaces. Any Tate vector space can be represented as a direct sum of a discrete and a compact space.

A morphism of Tate vector spaces $A : F \to G$ is *compact* if the closure of $\mathrm{Im}A$ is compact; it is *discrete* if $\mathrm{Ker}A$ is open. The composition of a compact operator with

[94] Here the superscript "cl" means "classical", see 3.8.8 for an explanation.

[95] i.e., the topology has a base formed by vector subspaces of F.

[96] Tate vector spaces, as well as their key attribute – the Tate extension of $\mathfrak{g}l$ – appear in [T] implicitly. In fact, they were introduced by Lefschetz ([Lef], p. 78–79) under the name of locally linearly compact spaces and used by Chevalley (these authors did not conceive, however, the Tate extension). So a proper name would be LCT vector space (which may be also understood as "locally compact topological").

[97] := the space of all continuous linear functionals $F \to k$; open subspaces in F^* are orthogonal complements to compact subspaces of F.

[98] i.e., a vector space equipped with a discrete topology.

an arbitrary one is compact; same for discrete operators. A is compact if and only if $A^* : G^* \to F^*$ is discrete. We denote by $\mathrm{Hom}_+(F,G), \mathrm{Hom}_-(F,G) \subset \mathrm{Hom}(F,G)$ the subspaces of compact, resp. discrete, operators, and by $\mathrm{Hom}_0(F,G)$ the intersection of these subspaces. Any operator may be represented as sum of a compact and a discrete operator.

The above vector spaces of operators carry natural topologies. Notation: for subspaces $U \subset F$ and $V \subset G$ set $\mathrm{Hom}(F,G)_{U,V} := \{A \in \mathrm{Hom}(F,G) : A(U) \subset V\}$. Now bases of topologies on $\mathrm{Hom}(F,G)$, $\mathrm{Hom}_+(F,G)$, $\mathrm{Hom}_-(F,G)$, and $\mathrm{Hom}_0(F,G)$ are formed, respectively, by subspaces $\mathrm{Hom}(F,G)_{U,V}$, $\mathrm{Hom}(F,V)$, $\mathrm{Hom}(F/U,G)$, and $\mathrm{Hom}(F/U,V)$ where $U \subset F$ and $V \subset G$ are c-lattices. These topologies are complete and separated. The exact sequence

$$0 \to \mathrm{Hom}_0(F,G) \xrightarrow{\alpha} \mathrm{Hom}_+(F,G) \oplus \mathrm{Hom}_-(F,G) \xrightarrow{\beta} \mathrm{Hom}(F,G) \to 0, \tag{2.6.9.1}$$

where $\alpha(A) := (-A, A)$, $\beta(A,B) := A + B$, is strongly compatible with the topologies.[99]

Notice that $\mathrm{Hom}_\pm$ and Hom_0 are two-sided ideals in the obvious sense, and all the composition maps like $\mathrm{Hom}_-(G,H) \times \mathrm{Hom}_+(F,G) \to \mathrm{Hom}_0(F,H)$, etc., are continuous. The maps $\mathrm{Hom}(F,G) \xrightarrow{\sim} \mathrm{Hom}(G^*,F^*)$, $\mathrm{Hom}(F,G)_+ \xrightarrow{\sim} \mathrm{Hom}(G^*,F^*)_-$, etc., which send A to A^*, are homeomorphisms.

Exercise. Consider an (abstract) vector space $G \otimes F$. It carries 4 natural topologies with bases of opens formed, respectively, by subspaces $V \otimes F + G \otimes U$, $V \otimes F$, $G \otimes U$, and $V \otimes U$, where $V \subset G$, $U \subset F$ are c-lattices. Now replace F by F^* and consider the obvious canonical map $G \otimes F^* \to \mathrm{Hom}_0(F,G)$. Show that it yields homeomorphisms between the completions of $G \otimes F^*$ with respect to these topologies and, respectively, $\mathrm{Hom}(F,G)$, $\mathrm{Hom}_+(F,G)$, $\mathrm{Hom}_-(F,G)$, and $\mathrm{Hom}_0(F,G)$.

So for a Tate space F we have an associative algebra $\mathrm{End}F$ together with two-sided ideals $\mathrm{End}_0F, \mathrm{End}_\pm F \subset \mathrm{End}F$ such that $\mathrm{End}_+F + \mathrm{End}_-F = \mathrm{End}F$. An operator A belongs to $\mathrm{End}_0F := \mathrm{End}_+F \cap \mathrm{End}_-F$ if and only if there are c-lattices $V \subset U$ such

[99] i.e., α is a homeomorphism onto a closed subspace, β is continuous and open.

that $A(V) = 0$, $A(F) \subset U$. For such A its *trace* $\mathrm{tr}A$ is the trace of the induced operator on U/V; it does not depend on the auxiliary choice of lattices. For any $A_\pm \in \mathrm{End}_\pm F$ one has $\mathrm{tr}([A_+, A_-]) = 0$. Since $\mathrm{End}F = \mathrm{End}_+F + \mathrm{End}_-F$ one has $\mathrm{tr}[A, B] = 0$ for any $A \in \mathrm{End}_0 F$, $B \in \mathrm{End}F$. The functional tr is continuous with respect to our topology on $\mathrm{End}_0 F$. In terms of the above exercise, tr is simply the continuous extension of the canonical pairing $F \otimes F^* \to k$.

2.6.10 The Tate extension in the linear algebra setting. For a Tate vector space F let $\mathfrak{g}l(F)$ be $\mathrm{End}F$ considered as a Lie algebra. By 2.6.9 we have ideals $\mathfrak{g}l_0(F) \subset \mathfrak{g}l_\pm(F) \subset \mathfrak{g}l(F)$ and $\mathrm{tr} : \mathfrak{g}l_0(F) \to k$ that fit into the setting of 2.6.1(ii). The corresponding central extension $\mathfrak{g}l(F)^\natural$ is called the *Tate extension*. It satisfies the cyclic property (see 2.6.1(iii)). We have canonical sections $s_\pm = s_\pm^F : \mathfrak{g}l_\pm(F) \to \mathfrak{g}l(F)^\natural$.

According to 2.6.9 the Lie algebras $\mathfrak{g}l(F)$, $\mathfrak{g}l_\pm(F)$, $\mathfrak{g}l_0(F)$ carry canonical topologies, and the functional tr is continuous. Our $\mathfrak{g}l(F)^\natural$ is also a topological Lie algebra: the topology is defined by the condition that the map $s_+ + s_- : \mathfrak{g}l_+(F) \times \mathfrak{g}l_-(F) \to \mathfrak{g}l(F)^\natural$ is continuous and open. It is complete and separated; the projection $\mathfrak{g}l(F)^\natural \to \mathfrak{g}l(F)$ is continuous and open.

Remark. The identification $\mathfrak{g}l(F) \xrightarrow{\sim} \mathfrak{g}l(F^*)$, $A \mapsto -A^*$, lifts canonically to an identification of topological k-extensions $\mathfrak{g}l(F)^\natural \xrightarrow{\sim} \mathfrak{g}l(F^*)^\natural$ which interchanges s_+ and s_-.

The Tate extension satisfies an important additivity property. Let $\mathcal{F}$ be a finite flag $F_1 \subset F_2 \subset ... \subset F_n = F$ of closed subspaces of F.[100] Then F_i are Tate vector spaces, as well as $\mathrm{gr}_i F$. Let $\mathfrak{g}l(F)_\mathcal{F} \subset \mathfrak{g}l(F)$ be the stabilizer of our flag, so one has the projections $\pi_i : \mathfrak{g}l(F)_\mathcal{F} \to \mathfrak{g}l(\mathrm{gr}_i F)$. There are two natural k-extensions $\mathfrak{g}l(F)^\natural_\mathcal{F}$ and $\mathfrak{g}l(F)^{\natural'}_\mathcal{F}$ of $\mathfrak{g}l(F)_\mathcal{F}$: the first one is the restriction of $\mathfrak{g}l(F)^\natural$ to $\mathfrak{g}l(F)_\mathcal{F}$, the second one is the Baer sum of the π_i-pull-backs of the extensions $\mathfrak{g}l(\mathrm{gr}_i F)^\natural$.

2.6.11 Lemma. The topological central extensions $\mathfrak{g}l(F)^\natural_\mathcal{F}$, $\mathfrak{g}l(F)^{\natural'}_\mathcal{F}$ are canonically isomorphic.

[100] Any such flag splits, i.e., it comes from a grading.

Proof. Consider the ideals $\mathfrak{g}l_\pm(F)_\mathcal{F} := \mathfrak{g}l(F)_\mathcal{F} \cap \mathfrak{g}l_\pm(F)$; one has $\mathfrak{g}l_+(F)_\mathcal{F} + \mathfrak{g}l_-(F)_\mathcal{F} = \mathfrak{g}l(F)_\mathcal{F}$ and $\pi_i(\mathfrak{g}l_\pm(F)_\mathcal{F}) \subset \mathfrak{g}l_\pm(\mathrm{gr}_i F)$. Both $\mathfrak{g}l(F)^\flat_\mathcal{F}$ and $\mathfrak{g}l(F)^{\flat'}_\mathcal{F}$ are equipped with sections over $\mathfrak{g}l_\pm(F)_\mathcal{F}$: these are, respectively, $s_\pm := s^F_\pm$ restricted to $\mathfrak{g}l_\pm(F)_\mathcal{F}$ and $s'_\pm := \Sigma\pi_i^*(s_\pm^{\mathrm{gr}_i F})$. Over $\mathfrak{g}l_0(F)_\mathcal{F} := \mathfrak{g}l_+(F)_\mathcal{F} \cap \mathfrak{g}l_-(F)_\mathcal{F}$ one has $s_+ - s_- = s'_+ - s'_- = \mathrm{tr}$. Now our lemma follows from the uniqueness statement of 2.6.1(ii). □

Remark. Assume that in the situation of the above lemma every $\mathrm{gr}_i F$ is either compact or discrete. Then the corresponding sections $s_\pm^{\mathrm{gr}_i F}$ (together with the lemma) provide a splitting of $\mathfrak{g}l(F)^\flat_\mathcal{F}$. For example, if $\mathcal{F}$ consists of a single c-lattice $P \subset F$ then s^P_+ and $s^{F/P}_-$ yield a canonical splitting s_P of $\mathfrak{g}l(F)^\flat_P$. If $P' \subset P$ is another c-lattice then on $\mathfrak{g}l(F)_P \cap \mathfrak{g}l(F)_{P'}$ one has $s_P - s_{P'} = \mathrm{tr}_{P/P'}$.

2.6.12. We return to the geometric setting of 2.6.2. Let $x \in X$ be a (closed) point, $j_x : U_x \hookrightarrow X$ the complement to x, $O_x \subset K_x$ is the completion of the local ring at x and its field of fractions. Let V be a coherent $\mathcal{D}$-module on U_x. Set $V_{(x)} := \hat{h}_x(j_{x*}V)$ (see 2.1.13). This is a Tate vector space (see Remark in 2.1.14). Applying 2.1.14(ii) to $M = j_{x*}V$ we get a canonical isomorphism of topological vector spaces

$$V \underset{\mathcal{D}_{U_x}}{\otimes} K_x \xrightarrow{\sim} V_{(x)}. \tag{2.6.12.1}$$

Here $V \underset{\mathcal{D}_{U_x}}{\otimes} K_x := \Gamma(U_x, V) \underset{\Gamma(U_x,\mathcal{D})}{\otimes} K_x = V_\eta \underset{\mathcal{D}_\eta}{\otimes} K_x$ (where η is the generic point of X) and its topology is formed by the images of $V_\xi \underset{\mathcal{D}_X}{\otimes} O_x$, $V_\xi \in \Xi_x(j_{x*}V)$ (see 2.1.14(i)).

We assume from now on that V is a $\mathcal{D}$-vector bundle on U_x;[101] let V' be its dual. The $*$ pairing $j_{x*}V \boxtimes j_{x*}V' \to \Delta_* j_{x*}\omega_{U_x}$ composed with the canonical projection $j_{x*}\omega_{U_x} \twoheadrightarrow i_{x*}k$ yields a canonical k-valued pairing between the Tate vector spaces $V_{(x)}$ and $V'_{(x)}$. We claim that this pairing is non-degenerate, so $V_{(x)}$ and $V'_{(x)}$ are mutually dual Tate vector spaces. To prove this it is enough to consider the following example:

Example. If $V = \mathcal{D}_{U_x}$ then $V' = \omega_{U_x} \otimes \mathcal{D}_{U_x}$. One has $V_{(x)} = K_x$, $V'_{(x)} = \omega_{K_x}$, and the above pairing is $f, \nu \mapsto \mathrm{Res}_x(f\nu)$.

[101] In fact, V becomes a free $\mathcal{D}$-module after replacing X by some neighbourhood of x. This follows from the fact that every ideal of $\mathcal{D}_\eta$ is principal (here η is the generic point of X).

2.6.13. Consider now the Lie* algebra $j_{x*}\mathfrak{g}l(V)$. It acts on $j_{x*}V$ and $j_{x*}V'$.

Assume we have $V_\xi \in \Xi_x(j_{x*}V)$, $V'_{\xi'} \in \Xi_x(j_{x*}V')$ such that the canonical $*$ pairing sends $V_\xi \boxtimes V'_{\xi'}$ to $\Delta_*\omega_X \subset \Delta_* j_{x*}\omega_{U_x}$. Then the matrix algebra $V_\xi \otimes V'_{\xi'}$ is a Lie* subalgebra of $j_{x*}\mathfrak{g}l(V)$ which equals to $\mathfrak{g}l(V)$ on U_x. We call such a subalgebra *special*. Special Lie* subalgebras form a base of a topology Ξ_x^{sp} on $j_{x*}\mathfrak{g}l(V)$ at x (called *the special topology*) which is weaker than the topology Ξ_x^{Lie} (see 2.5.12).[102] It is clear that the action of $j_{x*}\mathfrak{g}l(V)$ on $j_{x*}V$ is continuous with respect to the Ξ_x^{sp}- and Ξ_x-topologies.[103]

Denote by $\mathfrak{g}l(V)_{(x)}$ the completion of the Lie algebra $h_x(j_{x*}\mathfrak{g}l(V))$ with respect to Ξ_x^{sp}-topology (see 2.5.12 and Remark in 2.5.13). This is a topological Lie algebra[104] which acts continuously on the Tate vector space $V_{(x)}$.

2.6.14 Lemma. This action yields an isomorphism of topological Lie algebras

$$r : \mathfrak{g}l(V)_{(x)} \xrightarrow{\sim} \mathfrak{g}l(V_{(x)}). \tag{2.6.14.1}$$

Proof. It suffices to consider the case $V = \mathcal{D}_{U_x}$. Then $h_x(j_{x*}\mathfrak{g}l(V))$ is the vector space $\mathcal{D}_\eta$ of all meromorphic differential operators. The special topology has a base formed by the subspaces $\mathcal{D}_{m,n} := \{\partial : \partial(t^{-m}\mathcal{O}_x) \subset t^n\mathcal{O}_x\}$; here t is a parameter at x. This is exactly the topology induced from the topology on $\mathfrak{g}l(V_{(x)})$ (see 2.6.10, 2.6.9). It remains to show that $\mathcal{D}_\eta/\mathcal{D}_{m,n} \xrightarrow{\sim} \mathrm{Hom}(t^{-m}O_x, K_x/t^nO_x)$ (where Hom means continuous linear maps); this is a refreshing exercise for the reader. □

2.6.15. Now consider the central extension $j_{x*}\mathfrak{g}l(V)^\flat$ of $j_{x*}\mathfrak{g}l(V)$ by $j_{x*}\omega_{U_x}$. Notice that for any special subalgebra $\mathcal{G} = V_\xi \otimes V'_{\xi'} \subset j_{x*}\mathfrak{g}l(V)$ (see 2.6.13) its Tate extension[105] $\mathcal{G}^\flat$ is a Lie* subalgebra of $j_{x*}\mathfrak{g}l(V)^\flat$. We call such Lie* subalgebra of $j_{x*}\mathfrak{g}l(V)^\flat$ *special*. Special subalgebras form a base of a topology $\Xi_x^{sp\flat}$ on $j_{x*}\mathfrak{g}l(V)^\flat$ at x (called *the special topology*) which is weaker then the Ξ_x^{Lie}-topology (see 2.5.12).

---

[102] In fact, as follows from 2.5.14 and 2.6.14 the special topology is stricty weaker than Ξ_x^{Lie}.

[103] In fact, Ξ_x^{sp} is the weakest topology on $j_{x*}\mathfrak{g}l(V)$ such that the $j_{x*}\mathfrak{g}l(V)$-action on $j_{x*}V$ equipped with Ξ_x-topology is continuous.

[104] In fact, this is a topological associative algebra.

[105] With respect to the $*$ pairing $\in P_2^*(\{V_\xi, V'_{\xi'}\}, \omega_X)$.

Let $\mathfrak{g}l(V)^\flat_{(x)}$ be the completion of $h_x(j_{x*}\mathfrak{g}l(V)^\flat)$ with respect to $\Xi_x^{sp\flat}$-topology. The adjoint action of $j_{x*}\mathfrak{g}l(V)$ on $j_{x*}\mathfrak{g}l(V)^\flat$ is continuous with respect to special topologies, so $\mathfrak{g}l(V)^\flat_{(x)}$ is a topological Lie algebra. For every special subalgebra $\mathcal{G}^\flat \subset j_{x*}\mathfrak{g}l(V)^\flat$ we have $\mathcal{G}^\flat \cap j_{x*}\omega_{U_x} = \omega_X$, so the identification $\mathrm{Res}_x : j_{x*}\omega_{U_x}/\omega_X \xrightarrow{\sim} k$ tells us that $\mathfrak{g}l(V)^\flat_{(x)}$ is a topological central extension of $\mathfrak{g}l(V)_{(x)}$ by k.

2.6.16 Proposition. The isomorphism r of 2.6.14 lifts canonically to an isomorphism of topological central extensions

$$r^\flat : \mathfrak{g}l(V)^\flat_{(x)} \xrightarrow{\sim} \mathfrak{g}l(V_{(x)})^\flat. \tag{2.6.16.1}$$

Notice that such lifting $r^\flat$ is unique since the extension $\mathfrak{g}l(V_{(x)})^\flat$ has no automorphisms.

We give two constructions of $r^\flat$. The one right below is a version of considerations from [BS]. A more conceptual and simple approach uses chiral Clifford algebras; it can be found in 3.8.23.

Proof. (i) Let us compare the constructions of Tate extensions:

(a) Let η be the generic point of X. Apply the functor h to (2.6.2.2) and consider sections of the resulting sheaves over $\eta \times \eta$. We get an extension of $\bar{\mathfrak{g}}l(V)_\eta$-modules[106]

$$0 \to h(V)_\eta \otimes h(V')_\eta \to \bar{\mathfrak{g}}l(V)^\natural_\eta \to \bar{\mathfrak{g}}l(V)_\eta \to 0. \tag{2.6.16.2}$$

Consider the push-forward of this extension by the map $h(V)_\eta \otimes h(V')_\eta \xrightarrow{(\ )} h(\omega)_\eta \xrightarrow{\mathrm{Res}_x} k$ (see 2.6.2). This is a central extension of $\bar{\mathfrak{g}}l(V)_\eta$ by k; the bracket comes from the $\bar{\mathfrak{g}}l(V)_\eta$-module structure on $\bar{\mathfrak{g}}l(V)^\natural_\eta$. Our $\mathfrak{g}l(V)^\flat_{(x)}$ is the completion of this Lie algebra with respect to the topology defined by the images of vector spaces $h_{(x,x)}(j_*j^*V_\xi \boxtimes V'_{\xi'})$ where V_ξ,$V'_{\xi'}$ are as in 2.6.13.

(b) Look at (2.6.9.1) for $F = G = V_{(x)}$. This is an extension

$$0 \to \mathfrak{g}l_0(V_{(x)}) \xrightarrow{\alpha} \mathfrak{g}l(V_{(x)})^\natural \xrightarrow{\beta} \mathfrak{g}l(V_{(x)}) \to 0 \tag{2.6.16.3}$$

of topological $\mathfrak{g}l(V_{(x)})$-modules; here $\mathfrak{g}l(V_{(x)})^\natural := \mathfrak{g}l_+(V_{(x)}) \oplus \mathfrak{g}l_-(V_{(x)})$ and $\alpha = (+,-)$, $\beta = (+,+)$. Now $\mathfrak{g}l(V_{(x)})^\flat$ is the push-forward of this extension by $\mathrm{tr} : \mathfrak{g}l_0(V_{(x)}) \to k$; the bracket comes from the $\mathfrak{g}l(V_{(x)})$-module structure on $\mathfrak{g}l(V_{(x)})^\natural$.

[106] We use the notation from 2.6.4.

(ii) The action of $\bar{\mathfrak{g}l}(V)_\eta$ on the completion $V_{(x)}$ of $h(V)_\eta$ yields the morphism $r : \bar{\mathfrak{g}l}(V)_\eta \to \mathfrak{g}l(V_{(x)})$ of Lie algebras. We will lift it to a continuous morphism of extensions

$$r^\natural : \bar{\mathfrak{g}l}(V)^\natural_\eta \to \mathfrak{g}l(V_{(x)})^\natural \tag{2.6.16.4}$$

such that $r^\natural$ is an r-morphism of $\mathfrak{g}l(V)_\eta$-modules and its restriction to $h(V)_\eta \otimes h(V')_\eta$ equals the obvious morphism $h(V)_\eta \otimes h(V')_\eta \to V_{(x)} \otimes V'_{(x)} \to \mathfrak{g}l_0(V_{(x)})$. It is clear from (a), (b) above[107] that such $r^\natural$ yields the desired $r^\flat$.

(iii) Take $u^\natural \in \bar{\mathfrak{g}l}(V)^\natural_\eta = hj_*j^*(V \boxtimes V')_{\eta\times\eta}$; let $u \in \bar{\mathfrak{g}l}(V)_\eta$ be the image of $u^\natural$. To define $r^\natural(u^\natural) \in \mathfrak{g}l(V_{(x)})^\natural = \mathfrak{g}l_+(V_{(x)}) \oplus \mathfrak{g}l_-(V_{(x)})$ we must represent $r(u) \in \mathfrak{g}l(V_{(x)})$ as $r_+(u^\natural) + r_-(u^\natural)$, $r_\pm(u^\natural) \in \mathfrak{g}l_\pm(V_{(x)})$. Notice that $r(u) : h(V)_\eta \to h(V)_\eta$ equals $R_\Delta \circ f_{u^\natural}$ where $f_{u^\natural} : h(V)_\eta \to hj_*j^*(V \boxtimes \omega_X)_{\eta\times\eta}$ maps $s \in h(V)_\eta$ to $(id_V \boxtimes s)(u^\natural)$ (here we consider s as a morphism of $\mathcal{D}_X$-modules $V' \to \omega_X$ at η) and $R_\Delta : hj_*j^*(V \boxtimes \omega_X)_{\eta\times\eta} \to h(V)_\eta$ is the residue at $\eta \times x$ along the second variable. We have $R_\Delta = -R_x + \int_\eta$ where $\int_\eta$ denotes integration over the second variable and R_x is the residue at $\eta \times x$ along the second variable. It remains to prove the following lemma which is left to the reader:

Lemma. (a) The map $\int_\eta \circ f_{u^\natural} : h(V)_\eta \to h(V)_\eta$ extends to a compact operator $r^\natural_+(u^\natural) : V_{(x)} \to V_{(x)}$.

(b) The map $-R_x \circ f_{u^\natural} : h(V)_\eta \to h(V)_\eta$ extends to a discrete operator $r^\natural_-(u^\natural) : V_{(x)} \to V_{(x)}$.

(c) The map $r^\natural : \bar{\mathfrak{g}l}(V)^\natural_\eta \to \mathfrak{g}l(V_{(x)})^\natural$ defined by $r^\natural := (r^\natural_+, r^\natural_-)$ satisfies the properties promised in (ii). □

§7 Coisson algebras

2.7.1. According to 1.4.15 we have the category $\mathrm{Cois}(X) = \mathrm{Cois}(\mathcal{M}^{*!}(X))$ of coisson algebras on X. Explicitly, a coisson algebra is a commutative $\mathcal{D}_X$-algebra A^ℓ together with a Lie* bracket $\{\ \ \}$ on A (the coisson bracket) such that for any local sections $a, b, c \in A^\ell$ and $\nu \in \omega_X$ one has $\{ab\nu, c\nu\} = p_1^*a \cdot \{b\nu, c\nu\} + p_1^*b \cdot \{a\nu, c\nu\} \in \Delta_*A$ (we use the left $p_1^*A^\ell$-module structure on Δ_*A).

[107] Use the interpretation of tr at the end of 2.6.9.

The coisson brackets on a $\mathcal{D}_X$-algebra A^ℓ are étale local objects. Therefore we have the notion of coisson structure on any algebraic $\mathcal{D}_X$-space.

Let $x \in X$ be a (closed) point, $j_x: U_x := X \setminus \{x\} \hookrightarrow X$ the complementary open. Let A^ℓ be a coisson algebra on U_x. Consider the commutative topological algebra A_x^{as} defined in 2.4.8. Recall that A_x^{as} is the completion of the vector space $h(j_{x*}A)_x$ with respect to the Ξ_x^{as}-topology. The coisson bracket defines a Lie algebra structure on $h(j_{x*}A)_x$. The following lemma, as well as its proof, is parallel to 2.5.11:

2.7.2 Lemma. (i) The Lie bracket is Ξ_x^{as}-continuous with respect to each variable.

(ii) If A^ℓ is a countably generated $\mathcal{D}_{U_x}$-algebra then the Lie bracket is Ξ_x^{as}-continuous. Its continuous extension to A_x^{as} is a Poisson bracket.

Proof. (i) We want to show that for every $\bar{l} \in h(j_{x*}A)$ the operator $ad_{\bar{l}}$ is Ξ_x^{as}-continuous. By 2.2.17(i) it is Ξ_x-continuous, hence for every $A_\xi \in \Xi_x^{as}(A)$ its preimage $ad_{\bar{l}}^{-1}(A_\xi)$ is Ξ_x-open. Since $ad_{\bar{l}}$ is a derivation, $ad_{\bar{l}}^{-1}(A_\xi) \cap A_\xi \in \Xi_x^{as}$, q.e.d.

(ii) We will find for every $A_\xi \in \Xi_x^{as}(A)$ some $A_\zeta \in \Xi_x^{as}(A)$ such that $\{A_\zeta, A_\zeta\} \subset A_\xi$. Together with (i) it proves the Ξ_x^{as}-continuity of the bracket. Let $L \subset A_\xi$ be a countably generated $\mathcal{D}$-module such that its restriction to U_x generates A^ℓ as a $\mathcal{D}_{U_x}$-algebra. According to 2.2.17(ii) the restriction of the coisson bracket $h_x(L) \otimes h_x(L) \to h_x(j_{x*}A)$ is Ξ_x-continuous. So for some $L' \in \Xi_x(L)$ one has $\{h_x(L'), h_x(L')\} \subset h_x(A_\xi)$. Now take for A_ζ the subalgebra generated by L'. □

Assume that A satisfies the countability condition of 2.7.2(ii). We define the bracket on A_x^{as} extending the bracket on $h_x(j_{x*}A)$ by continuity. It is automatically a Poisson bracket since $h(j_{x*}A)$ acts on $j_{x*}A$ by derivations.

2.7.3. Let A^ℓ be an arbitrary (not necessary countably generated) coisson algebra. Denote by $\Xi_x^{\mathcal{P}ois}$ the topology at x formed by all $A_\xi \in \Xi_x^{as}(A)$ which are coisson subalgebras of $j_{x*}A$. The coisson bracket on $h_x(j_{x*}A)$ is continuous with respect to this topology,[108] so the corresponding completion is a topological Poisson algebra.

[108] Indeed, for every $A_\xi \in \Xi_x^{\mathcal{P}ois}(A)$ the action of the Lie algebra $h_x(A_\xi)$ on $A_{\xi x}^\ell$ is continuous with respect to the topology on $h_x(A_\xi)$ formed by $h_x(A_\zeta)$ where $A_\zeta \in \Xi_x^{\mathcal{P}ois}(A)$, $A_\zeta \subset A_\xi$.

If we are in situation of 2.7.2 then $A_x^{\mathcal{P}ois}$ is the completion of A_x^{as} with respect to topology formed by those open ideals which are Lie subalgebras of A_x^{as}. It seems (though we do not know an example) that A_x^{Pois} can be a small quotient of A_x^{as}.

2.7.4. ¿From now on we assume that X is a curve.

Let $S \subset X$ be a finite subset, $j_S : U_S \hookrightarrow X$ the complementary open embedding. Let A be a finitely generated $\mathcal{D}_{U_S}$-algebra equipped with a coisson structure, so, by 2.7.2, $A_S^{as} = \widehat{\bigotimes}_{s\in S} A_s^{as}$ (see 2.4.10) is a topological Poisson algebra. The canonical morphism $\Gamma(U_S, h(A)) \to A_S^{as}$ from 2.4.10 commutes with brackets, i.e., it is a Hamiltonian action of the Lie algebra $\Gamma(U_S, h(A))$ on A_S^{as}. If X is compact, then, according to 2.4.5, 2.4.10, the zero fiber of the momentum map coincides with $\operatorname{Spec}\langle A\rangle(U) \subset \operatorname{Spf} A_S^{as} = \prod \operatorname{Spf} A_s^{as}$.

2.7.5. Let $\mathcal{Y}$ be a smooth $\mathcal{D}_X$-scheme. A coisson bracket $\{\ \ \}$ on $\mathcal{Y}$ defines a Lie* algebroid structure on $\Omega_{\mathcal{Y}}$ (see 1.4.15), so we have the action map $\tau = \tau_{\{\ \}} \colon \Omega_{\mathcal{Y}} \to \Theta_{\mathcal{Y}}$. One says that $\{\ \ \}$ is *symplectic* if $\tau_{\{\ \}}$ is an isomorphism. This situation occurs fairly rarely. A better notion is *elliptic* coisson structure which we are going to define. We assume that $\dim X = 1$.

2.7.6 Definition. A Lie* algebroid $\mathcal{L}$ on $\mathcal{Y}$ is *elliptic* if the action map $\tau_{\mathcal{L}} \colon \mathcal{L} \to \Theta_{\mathcal{Y}}$ is injective and $\Phi_{\mathcal{L}} := \operatorname{Coker}(\tau_{\mathcal{L}})$ is a locally free $\mathcal{O}_{\mathcal{Y}}$-module of finite rank. A coisson structure on $\mathcal{Y}$ is *elliptic* if the corresponding Lie* algebroid $\Omega_{\mathcal{Y}}$ is elliptic. □

§8 Other settings

We define a canonical compound tensor structure on the categories of modules over a Lie algebroid and of Harish–Chandra modules.

2.8.1. The compound tensor structure on the category of $\mathcal{D}$-modules is a particular case of a canonical compound tensor structure on the category of modules over Lie algebroid.[109] Let us recall the basic definitions (for details see, e.g., [M] or first two sections of [BB]).

Let X be any k-scheme, $\mathcal{L}$ a Lie algebroid on X. This means that $\mathcal{L}$ is a sheaf of k-vector spaces on X equipped with Lie algebra and $\mathcal{O}_X$-module structures, and we also

[109] Subject to a condition from 2.8.2.

have an action τ of the Lie algebra $\mathcal{L}$ on $\mathcal{O}_X$.[110] We demand that for $f, g \in \mathcal{O}_X$, $l, l' \in \mathcal{L}$ one has $\tau(fl)g = f(\tau(l)g)$ and $[l, fl'] = \tau(l)(f)l' + f[l, l']$. We also assume that $\mathcal{L}$ is a quasicoherent $\mathcal{O}_X$-module.

Below we denote by $\mathcal{L}^-$ our $\mathcal{L}$ considered as a plain sheaf of Lie algebras equipped with an action τ on $\mathcal{O}_X$.[111]

An $\mathcal{L}^-$-*module* is a quasicoherent $\mathcal{O}_X$-module M equipped with an action of $\mathcal{L}^-$ which is compatible with the $\mathcal{L}^-$-action on $\mathcal{O}_X$, i.e., for every $f \in \mathcal{O}_X$, $m \in M$, and $l \in \mathcal{L}$ one has $l(fm) = l(f)m + f(lm)$. We say that M is a *left* (resp. *right*) $\mathcal{L}$-*module* if, in addition, one has $(fl)m = f(lm)$ (resp. $(fl)n = l(fn)$).

The category $\mathcal{M}(X, \mathcal{L}^-)$ of $\mathcal{L}^-$-modules is an abelian k-category. The $\mathcal{O}_X$-tensor product makes it a tensor category. The full subcategories $\mathcal{M}^\ell(X, L)$, $\mathcal{M}^r(X, \mathcal{L})$ of left and right, $\mathcal{L}$-modules are closed under subquotients. The tensor product of two left $\mathcal{L}$-modules is a left $\mathcal{L}$-module, and that of a left and a right $\mathcal{L}$-module is a right $\mathcal{L}$-module. Therefore $\mathcal{M}^\ell(X, \mathcal{L})$ is a tensor category, and $\mathcal{M}^r(X, \mathcal{L})$ is an $\mathcal{M}^\ell(X, \mathcal{L})$-Module.

Let $U(\mathcal{L})$ be the enveloping algebra of $\mathcal{L}$. This is a sheaf of associative algebras on X equiped with a morphism of algebras $i_{\mathcal{O}} : \mathcal{O}_X \to U(\mathcal{L})$ and a morphism of Lie algebras $i_{\mathcal{L}} : \mathcal{L} \to U(\mathcal{L})$ such that $i_{\mathcal{O}}$ is a morphism of Lie algebra $\mathcal{L}$-modules ($\mathcal{L}$ acts on $U(\mathcal{L})$ by $i_{\mathcal{L}}$ composed with the adjoint action) and $i_{\mathcal{L}}$ is a morphism of $\mathcal{O}_X$-modules ($\mathcal{O}_X$ acts on $U(\mathcal{L})$ by $i_{\mathcal{O}}$ and left multiplication), and $U(\mathcal{L})$ is universal with respect to this datum. Then left, resp. right, $\mathcal{L}$-modules are the same as left (right) $U(\mathcal{L})$-modules which are quasicoherent as $\mathcal{O}_X$-modules.

2.8.2. ¿From now on we assume that $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of finite rank n. The adjoint action of $\mathcal{L}$ on $\mathcal{L}$ makes it an $\mathcal{L}^-$-module. One checks in a moment that the line bundle $\nu_{\mathcal{L}} := \det(\mathcal{L})^*$ is a right $\mathcal{L}$-module. There is a canonical equivalence of categories

$$\mathcal{M}^\ell(X, \mathcal{L}) \xrightarrow{\sim} \mathcal{M}^r(X, \mathcal{L}), \qquad M \mapsto M^r := M \otimes \nu_{\mathcal{L}}. \tag{2.8.2.1}$$

We denote these categories thus identified by $\mathcal{M}(X, \mathcal{L})$.

[110] i.e., τ is a morphism of Lie algebras $\mathcal{L} \to \Theta_X := \mathrm{Der}(\mathcal{O}_X)$.

[111] i.e., we forget about the $\mathcal{O}$-module structure on $\mathcal{L}$.

Remark. Assume that our algebroid is rigidified, i.e., we have a Lie algebra L and a morphism of Lie algebras $L \to \mathcal{L}$ which yields an isomorphism $\mathcal{O}_X \underset{k}{\otimes} L \xrightarrow{\sim} \mathcal{L}$. Then L acts on $\mathcal{O}_X$, so we have the category $\mathcal{M}(X, L)$ of quasicoherent $\mathcal{O}_X$-modules equipped with an L-action compatible with the L-action on $\mathcal{O}_X$. Then the obvious "identity" functors $\mathcal{M}^\ell(X, \mathcal{L}) \to \mathcal{M}(X, L)$, $\mathcal{M}^r(X, \mathcal{L}) \to \mathcal{M}(X, L)$ are equivalences of categories. The corresponding composition equivalence $\mathcal{M}^\ell(X, \mathcal{L}) \xrightarrow{\sim} \mathcal{M}(X, \mathcal{L})$ differs from (2.8.2.1) by twist by $\det L$.

2.8.3. Let us define on $\mathcal{M}(X, \mathcal{L})$ the structure of compound tensor category. We already have the tensor structure. It remains to define the $*$ operations and the compatibility morphisms.

2.8.1. Recall that *a Harish–Chandra pair* $(\mathfrak{g}, K)$ consists of a Lie algebra $\mathfrak{g}$, an affine group scheme K together with an embedding $i : \mathfrak{k} := \mathrm{Lie}K \hookrightarrow \mathfrak{g}$ and a K-action on $\mathfrak{g}$

CHAPTER THREE: CHIRAL BASICS

Алгебраисты обычно определяют группы как множества с операциями, удовлетворяющими длинному ряду трудно-запоминаемых аксиом. Понять такое определение, на мой взгляд, невозможно.

Член-корреспондент АН СССР В. И. Арнольд, "Математика с человеческим лицом", Природа, №3, 1988, 117–118 †

§1 Chiral operations

¿From now on we assume that X is a smooth curve.

3.1.1. For a finite non-empty set I put $U^{(I)} := \{(x_i) \in X^I : x_{i_1} \neq x_{i_2} \text{ for every } i_1 \neq i_2\}$. So $U^{(I)}$ is the complement to the diagonal divizor if $|I| \geq 2$. Denote by $j^{(I)} : U^{(I)} \hookrightarrow X^I$ the open embedding. We write $U^{(n)}$ for $U^{(\{1,..,n\})}$.

Let L_i, $i \in I$, be an I-family of (right) $\mathcal{D}$-modules on X. For a $\mathcal{D}_X$-module M set

$$P_I^{ch}(\{L_i\}, M) := \mathrm{Hom}_{\mathcal{M}(X^I)}(j_*^{(I)} j^{(I)*}(\underset{I}{\boxtimes} L_i), \Delta_*^{(I)} M). \tag{3.1.1.1}$$

The elements of this vector space are called *chiral I-operations.* They have local nature with respect to the étale topology of X. As usual, we write P_n^{ch} for $P_{\{1,..,n\}}^{ch}$; the elements of P_2^{ch} are called *chiral pairings.*

Remark. Let F_i ,G be quasicoherent $\mathcal{O}_X$-modules. According to 2.1.7, 2.1.8(iii) we have $P_I^{ch}(\{F_{i\mathcal{D}}\}, G_{\mathcal{D}}) = \mathrm{Diff}(j_*^{(I)} j^{(I)*}(\boxtimes F_i), \Delta.G)$; elements of this vector space are called *chiral polydifferential operators.*

† Algebraists define a group as a set equipped with operations, subject to a long series of axioms which are difficult to remember. Understanding such a definition is, in my opinion, impossible. V. I. Arnold, Corresponding Member of the Academy of Sciences of the USSR, "Mathematics with a Human Face", Priroda, No. 3, 1988, 117–118.

3.1.2. Our P_I^{ch} are left exact k-polylinear functors on $\mathcal{M}(X)$. Let us define the composition of chiral operations. Let $\pi: J \twoheadrightarrow I$ be a surjective map of finite non-empty sets, $\{K_j\}$ a J-family of $\mathcal{D}_X$-modules. The composition map

$$P_I^{ch}(\{L_i\}, M) \otimes (\otimes_I P_{J_i}^{ch}(\{K_j\}, L_i)) \longrightarrow P_J^{ch}(\{K_j\}, M) \tag{3.1.2.1}$$

sends $\varphi \otimes (\otimes \psi_i)$ to $\varphi(\psi_i)$ defined as the composition

$$j_*^{(J)} j^{(J)*}(\boxtimes_J K_j) = j_*^{(\pi)} j^{(\pi)*}(\boxtimes_I (j_*^{(J_i)} j^{(J_i)*} \boxtimes_{J_i} K_j)) \xrightarrow{\boxtimes \psi_i} j_*^{(\pi)} j^{(\pi)*}(\boxtimes_I \Delta_*^{(J_i)} L_i)$$
$$= \Delta_*^{(\pi)} j_*^{(I)} j^{(I)*}(\boxtimes_I L_i) \xrightarrow{\Delta_*^{(\pi)}(\varphi)} \Delta_*^{(\pi)} \Delta_*^{(I)} M = \Delta_*^{(J)} M.$$

Here $j^{(\pi)}$: $U^{(\pi)} := \{(x_j) \in X^J : x_{j_1} \neq x_{j_2} \text{ if } \pi(j_1) \neq \pi(j_2)\} \hookrightarrow X^J$; for the rest of notation see 2.2.3.

The composition is associative, so P_I^{ch} define on $\mathcal{M}(X)$ an abelian pseudo-tensor structure on $\mathcal{M}(X)$. We call it the *chiral structure* and denote by $\mathcal{M}(X)^{ch}$. The pseudo-tensor categories $\mathcal{M}(U)^{ch}$, $U \in X_{\text{ét}}$, form a sheaf of pseudo-tensor categories $\mathcal{M}(X_{\text{ét}})^{ch}$ on the étale topology of X.

The de Rham functor $h = h_X$ (see 2.1.6) is an augmentation functor on $\mathcal{M}(X_{\text{ét}})^{ch}$ (see 1.2.5).[112] The structure map

$$h_{I,i_0}: P_I^{ch}(\{L_i\}, M) \otimes h(L_{i_0}) \to P_{I \smallsetminus \{i_0\}}^{ch}(\{L_i\}, M) \tag{3.1.2.2}$$

sends $\varphi \otimes \bar{\ell}_{i_0}$, where $\varphi \in P_I^{ch}(\{L_i\}, M)$, $\ell_{i_0} \in L_{i_0}$, to the chiral $I \smallsetminus \{i_0\}$-operation $a \mapsto Tr_{i_0} \varphi(a \boxtimes \ell_{i_0})$ (cf.2.2.7). Here, as in 2.2.7, Tr_{i_0}: $pr_{I,i_0}.\Delta_*^{(I)} M \to \Delta_*^{(I \smallsetminus \{i_0\})} M$ is the trace map for the projection pr_{I,i_0}: $X^I \to X^{I \smallsetminus \{i_0\}}$. Compatibilities 1.2.5(i),(ii) are immediate.

3.1.3. Consider the operad $P^{ch}(\omega_X)$ (see 1.1.6) of chiral operations acting on ω_X. One has $P^{ch}(\omega_X)_I := \mathrm{Hom}(j_*^{(I)} j^{(I)*} \omega_X^{\boxtimes I}, \Delta_*^{(I)} \omega_X)$. Set $\lambda_I := \det(k^I)$; this is a line on which Aut I acts by the sgn character. One has a canonical (in particular, Aut I-equivariant) isomorphism ε_I: $\omega_X^{\boxtimes I} \otimes \lambda_I \xrightarrow{\sim} \omega_{X^I}$, $(\boxtimes \nu_i) \otimes (e_{i_1} \wedge .. \wedge e_{i_n}) \mapsto pr_{i_1}^* \nu_{i_1} \wedge .. \wedge pr_{i_n}^* \nu_{i_n}$; here $\nu_i \in \omega_X$, $\{i_1, .., i_n\}$ is an ordering of I, and $\{e_i\}$ is the standard base of k^I. If $|I| = 2$ the residue Res: $j_*^{(I)} \omega_{U^{(I)}} \to \Delta_*^{(I)} \omega_X$ yields a canonical map r_I: $\lambda_I \to \mathrm{Hom}(j_*^{(I)} \omega_{U^{(I)}}, \Delta_*^{(I)} \omega_X) = P^{ch}(\omega_X)_I$.

[112] Contrary to the $*$ situation, in the chiral setting h is highly degenerate.

3.1.4 Proposition. There is a unique isomorphism of operads

$$\kappa : \mathcal{L}ie \xrightarrow{\sim} P^{ch}(\omega_X)$$

which coincides with r_I for $|I| = 2$.

Beginning of the proof. Our basic tool is the Cousin complex C_I for ω_{X^I} with respect to the diagonal stratification. One has $C_I^m := \bigoplus_{T \in Q(I,|I|-m)} \Delta_*^{(I/T)} j_*^{(T)} \omega_{U^{(T)}}$ where $Q(I,t)$ is the set of quotient sets $I \twoheadrightarrow T$, $|T| = t$ (see 1.3.1), and $\Delta^{(I/T)}$ is the diagonal embedding $X^T \hookrightarrow X^I$. The non-zero components of the differential $d^m : C_I^m \to C_I^{m+1}$ are the morphisms $\Delta_*^{(I/T)}(\mathrm{Res}_{T'}) : \Delta_*^{(I/T)} j_*^{(T)} \omega_{U^{(T)}} \to \Delta_*^{(I/T')} j_*^{(T')} \omega_{U^{(T')}}$ for $X^{T'} \subset X^T$ where $\mathrm{Res}_{T'}$ is the residue morphism. The complex C_I is a resolution of ω_{X^I}.

It is easy to prove the existence of κ: the map $r_I : \lambda_I \to P^{ch}(\omega_X)_I$, $|I| = 2$, is Aut$I$-equivariant, so it defines a skew-symmetric operation $[\ \ ] \in P^{ch}(\omega_X)_2$, and we have only to show that the operation $\alpha := [[1,2],3] + [[2,3]1] + [[3,1],2] \in P^{ch}(\omega_X)_3$ is zero. In fact, $\alpha = d^1 d^0$ where d^j are the differentials in the complex C_I for $I = \{1,2,3\}$. The uniqueness of κ is clear. We will show that κ is bijective in 3.1.5-3.1.8.

3.1.5. Let M be a $\mathcal{D}_{X^I}$-module. We say that a finite increasing filtration $W.$ on M is *special* if for every l the successive quotient $\mathrm{gr}_l^W M$ is a finite sum of copies of $\Delta_*^{(I/T)} \omega_{X^T}$ for $T \in Q(I,-l)$. Such a filtration is unique if it exists. We call M a *special* $\mathcal{D}_{X^I}$-module if it has a special filtration. Let $\mathcal{M}(X^I)^{sp} \subset \mathcal{M}(X^I)$ be the full subcategory of special $\mathcal{D}$-modules. It is closed under finite direct sums and subquotients, so it is an abelian category. Any morphism in $\mathcal{M}(X^I)^{sp}$ is strictly compatible with $W.$, i.e., gr^W is an exact functor.

3.1.6 Lemma. $j_*^{(I)} \omega_{U^{(I)}}$ is a special $\mathcal{D}_{X^I}$-module.

Proof. We proceed by induction with respect to $|I|$ using the exact sequence $0 \to \omega_{X^I} \to C_I^0 \xrightarrow{d^0} C_I^1 \xrightarrow{d^1} C_I^2$ (see 3.1.4). Since $C_I^m \in \mathcal{M}(X^I)^{sp}$ for $m > 0$ by the induction assumption, the $\mathcal{D}$-module $K := \mathrm{Ker}\, d^1$ is special. Since $j^{(I)*} K = 0$ one has $W_{-|I|} K = 0$, so the following filtration $W.$ on $j_*^{(I)} \omega_{U^{(I)}} = C_I^0$ is special: $W_l C_I^0 := (d^0)^{-1}(W_l K)$ for $l \geq -|I|$, $W_l C_I^0 = 0$ for $l < -|I|$. $\square$

3.1.7. So for any $T \in Q(I)$ the $\mathcal{D}_{X^I}$-module $\Delta_*^{(I/T)} j_*^{(T)} \omega_{U(T)}$ is special. In fact, this is an injective object of $\mathcal{M}(X^I)^{sp}$. This follows from exactness of the functor $P \mapsto P/W_{-|T|-1}P$ since $\Delta_*^{(I/T)} j_*^{(T)} \omega_{U(T)}$ is obviously an injective object of the subcategory $\mathcal{M}(X^I)^{sp}_{\geq -|T|} := \{P : W_{-|T|-1}P = 0\}$. In particular, $\Delta_*^{(I)} \omega_{U(I)}$ is an injective object of $\mathcal{M}(X^I)^{sp}$.

Assume that $|I| > 1$. Since $C_I^{\cdot} \in \mathcal{M}(X^I)^{sp}$ and $\mathrm{Hom}(\omega_{X^I}, \Delta_*^{(I)} \omega_X) = 0$, the complex $\mathrm{Hom}(C_I, \Delta_*^{(I)} \omega_X)$ is acyclic. Using the canonical isomorphism $\omega_{X^I} = \omega_X^{\boxtimes I} \otimes \lambda_I$ (see 3.1.3) we may rewrite this exact complex as follows:

$$.. \to \bigoplus_{T' \in Q(I, |I|-2)} P^{ch}(\omega_X)_{T'} \otimes \lambda_{T'} \xrightarrow{d_1} \bigoplus_{T \in Q(I, |I|-1)} P^{ch}(\omega_X)_T \otimes \lambda_T \xrightarrow{d_0} P^{ch}(\omega_X)_I \otimes \lambda_I \to 0.$$

Let us explain the operad meaning of the differential. Take $T' \in Q(T, |T|-1)$; let $\alpha', \alpha" \in T$ be the two distinct T'-equivalent elements of T, $\alpha \in T'$ the corresponding element of T'. Then $T \smallsetminus \{\alpha', \alpha''\} = T' \smallsetminus \{\alpha\}$, so we have a canonical identification

$$\lambda_{\{\alpha', \alpha''\}} \xrightarrow{\sim} \lambda_{T'} \otimes \lambda_T^{\otimes -1}, \quad e_{\alpha'} \wedge e_{\alpha''} \mapsto (\gamma \wedge e_\alpha) \otimes (\gamma \wedge e_{\alpha'} \wedge e_{\alpha''})^{-1}, \tag{3.1.7.1}$$

where γ is a generator of $\lambda_{T' \smallsetminus \{\alpha\}} = \lambda_{T \smallsetminus \{\alpha', \alpha''\}}$. If S is a set of two elements, then the above complex for $I = S$ reduces to the isomorphism $P^{ch}(\omega_X)_S = \lambda_S$. Therefore we may rewrite any non-zero component of d as the morphism

$$P^{ch}(\omega_X)_{T'} \otimes \lambda_{\{\alpha', \alpha''\}} = P^{ch}(\omega_X)_{T'} \otimes P^{ch}(\omega_X)_{\{\alpha', \alpha''\}} \to P^{ch}(\omega_X)_T.$$

This arrow coincides with the composition of operations map for $T \twoheadrightarrow T'$.

3.1.8. *End of the proof of* 3.1.4. The surjectivity of d_0 means that for $n \geq 2$ every operation in $P^{ch}(\omega_X)_n$ is a sum of $(n-1)$-operations composed with binary operations. Therefore $P^{ch}(\omega_X)$ is generated by $P^{ch}(\omega_X)_2$. Exactness of the complex from 3.1.7 means that all the relations in our operad come from the Jacobi relation for $[\quad] \in P^{ch}(\omega_X)_2$. $\square$

3.1.9 Remarks. (i) For $S, T \in Q(I)$ let $\mathcal{L}ie_{S/T}$ be the vector space $\bigotimes_T \mathcal{L}ie_{S_t}$ if $S \geq T$ and 0 otherwise. Let us define a canonical isomorphism

$$\alpha(I,S)_l\colon \ \mathrm{gr}^W_{-l}(\Delta_*^{(I/S)} j_*^{(S)} j^{(S)*} \omega_X^{\boxtimes S}) \xrightarrow{\sim} \bigoplus_{T \in Q(I,l)} \Delta_*^{(I/T)} \omega_X^{\boxtimes T} \otimes \mathcal{L}ie^*_{S/T}. \tag{3.1.9.1}$$

First notice that both parts are supported on $X^S \subset X^I$, so (applying $\Delta^{(I/S)!}$ and replacing S by I) we may assume that $S = I$. The case of top quotient $l = 1$ is just the isomorphism κ_I of 3.1.4. Assume that $l > 1$. It suffices to define $\alpha(I,I)_l = \alpha(I)_l$ over the complement to diagonals of dimension $< l$. Its T-component $\alpha(I)_T$ is determined by the restriction to a neighbourhood of X^T. There $j_*^{(I)} j^{(I)*} \omega_X^{\boxtimes I}$ coincides with $\underset{T}{\boxtimes} j_*^{(I_t)} j^{(I_t)*} \omega_X^{\boxtimes I_t}$, and $\alpha(I)_T := \underset{T}{\boxtimes} \alpha(I_t)_1$.

(ii) Let us describe $\mathcal{M}(X^I)^{sp}$ explicitly. Its irreducible objects are $\Delta_*^{(I/T)} \omega_X^{\boxtimes T}$, $T \in Q(I)$. The object $\mathcal{I}_T := \Delta_*^{(I/T)} j_*^{(T)} j^{(T)*} \omega_X^{\boxtimes T}$ is injective by 3.1.7; it contains $\Delta_*^{(I/T)} \omega_X^{\boxtimes T}$.[113] Isomorphism 3.1.9.1 yields a canonical identification $\mathrm{Hom}(\mathcal{I}_S, \mathcal{I}_T) = \mathcal{L}ie_{S/T}$. The composition of morphisms corresponds to (a tensor product) of compositions of Lie operations.

So let $\mathcal{Q}(I)$ be the k-category whose objects are elements of $Q(I)$ and $\mathrm{Hom}_{\mathcal{Q}(I)}(S,T) = \mathcal{L}ie_{S/T}$; the composition of morphisms is the tensor product of compositions of Lie operations. We have defined a fully faithful embedding $\mathcal{Q}(I) \to \mathcal{M}(X^I)^{sp}$, $T \mapsto \mathcal{I}_T$. A $\mathcal{Q}(I)$-*module* is a k-linear functor $\mathcal{Q}(I) \to$ (finite-dimensional vector spaces); $\mathcal{Q}(I)$-modules form an abelian category $\mathcal{V}(I)$. There is an obvious functor $\mathcal{M}(X^I)^{sp} \to \mathcal{V}(I)^\circ$ which sends a $\mathcal{D}$-module M to the $\mathcal{Q}(I)$-module $T \mapsto \mathrm{Hom}(M, \mathcal{I}_T)$. This functor is an equivalence of categories.

3.1.10 Remarks. We assume that $k = \mathbb{C}$.

(i) For any special $\mathcal{D}_{X^I}$-module P one has $H^l DR_{an}(P) = H^l DR_{an}(\mathrm{gr}_l^W P)$; this sheaf is isomorphic to a direct sum of constant sheaves supported on X^S, $|S| = -l$. Here DR_{an} denotes the analytic de Rham complex (used in the Riemann-Hilbert correwpondence). If $l = -1$ then $H^l DR_{an}(P)$ is the constant sheaf on the diagonal $X \subset X^I$ with fiber V^* where $V := \mathrm{Hom}(P, \Delta_*^{(I)} \omega_X)$.

(ii) The filtration $W.$ on $j_*^{(I)} \omega_{U^{(I)}}$ coincides with the weight filtration on the mixed Hodge module $j_*^{(I)} \mathbb{C}(|I|)_{U^{(I)}}[|I|]$.

(iii) By 3.1.3, one has $P^{ch}(\omega_X)_I = \mathrm{Hom}(j_*^{(I)} \omega_{U^{(I)}}, \Delta_*^{(I)} \omega_X) \otimes \lambda_I$. So remark (i) above implies that $P^{ch}(\omega_X)_I \otimes \lambda_I = H_{|I|-1}(Y_I, \mathbb{C})$ where Y_I is the configuration space of I-tuples

[113] $\mathcal{I}_T$ is the injective envelope of $\Delta_*^{(I/T)} \omega_X^{\boxtimes T}$ because $\dim \mathrm{Hom}(\Delta_*^{(I/T')} \omega_X^{\boxtimes T'}, \mathcal{I}_T)$ equals 0 if $T' \neq T$ and 1 if $T' = T$.

of different points on $\mathbb{C} = \mathbb{R}^2$. Up to homotopy the Y_I's are the spaces of the "small disc" operad. The composition law in the operad $P^{ch}(\omega_X)$ coincides with the one on the homology operad of this topological operad. So 3.1.4 amounts to Cohen's theorem [C].

3.1.11. For $M \in \mathcal{M}(X)$ we define *the unit operation* $\varepsilon_M \in P_2^{ch}(\{\omega_X, M\}, M)$ as the composition $j_*j^*\omega_X \boxtimes M \to (j_*j^*\omega_X \boxtimes M)/\omega_X \boxtimes M \xrightarrow{\sim} \Delta_* M$ where the last arrow comes from the canonical isomorphism $\omega_X \otimes^! M = M$.

3.1.12 Lemma. (i) For $M_i, N \in \mathcal{M}(X)$ there is a canonical isomorphism[114]

$$P_I^{ch}(\{M_i\}, N)^I \xrightarrow{\sim} P_{\tilde{I}}^{ch}(\{\omega_X, M_i\}, N), \quad (\varphi_i) \mapsto \sum_{i \in I} \varphi_i(\varepsilon_{M_i}, id_{M_{i'}})_{i' \neq i}.$$

(ii) Chiral operations satisfy Leibnitz rule with respect to the unit operations: for every $\varphi \in P_I^{ch}(\{M_i\}, N)$ one has $\varepsilon_N \varphi = \sum_{i \in I} \varphi(\varepsilon_{M_i}, id_{M_{i'}})_{i' \neq i} \in P_{\tilde{I}}^{ch}(\{\omega_X, M_i\}, N)$.

Proof. (i) Let $\Delta^i : X^I \hookrightarrow X^{\tilde{I}}$ be the i^{th} diagonal section of the projection $X^{\tilde{I}} \to X^I$. The images of the Δ^i's are disjoint over $U^{(I)}$, so one has a short exact sequence

$$(3.1.12.1) \quad 0 \to \omega_X \boxtimes j_*^{(I)} j^{(I)*}(\boxtimes M_i) \to j_*^{(\tilde{I})} j^{(\tilde{I})*} \omega_X \boxtimes (\boxtimes M_i) \to \oplus \Delta_*^i j_*^{(I)} j^{(I)*} \boxtimes M_i \to 0.$$

Here we use the standard identifications $j_*j^*\omega_X \boxtimes M_i/\omega_X \boxtimes M_i = \Delta_*(\omega_X \otimes^! M_i = \Delta_* M_i$. Since $\mathrm{Hom}(\omega_X \boxtimes j_*^{(I)} j^{(I)*}(\boxtimes M_i), \Delta_*^{(\tilde{I})} N) = 0$, (3.2.4.1) yields a canonical isomorphism $P_I^{ch}(\{M_i\}, N)^I \xrightarrow{\sim} P_{\tilde{I}}^{ch}(\{\omega_X, M_i\}, N)$. One checks in a moment that it coincides with the map of 3.1.12(i).

(ii) Let t be a local coordinate on X, t_0, t_i the corresponding local coordinates on $X^{\tilde{I}}$. For $i \in I$ set $\nu^i := \frac{dt_o}{t_0 - t_i}$. For any $m \in j_*^{(I)} j^{(I)*} \boxtimes M_i$ one has $\varepsilon_N \varphi(\nu^i \cdot m) = \Delta^i(\varphi(m)) = \varphi(\varepsilon_{M_i}, id_{M_{i'}})_{i' \neq i}(\nu_i \cdot m) \in \Delta_*^{(\tilde{I})} N$ and $\varphi(\varepsilon_{M_{i"}}, id_{M_{i'}})_{i' \neq i"} = 0$ for $i" \neq i$. Now use (i). $\square$

3.1.13 Variants. (i) Let L be any Lie* algebra, $M_i, N \in \mathcal{M}(X, L)$. A chiral operation $\varphi \in P_I^{ch}(\{M_i\}, N)$ is *L-compatible* if it satisfies the Leibnitz rule with respect to the L-actions. Explicitly, we demand that the composition $\cdot_N \varphi \in \mathrm{Hom}(L \boxtimes j_*^{(I)} j^{(I)*} \boxtimes M_i, \Delta_*^{(\tilde{I})} N)$

[114] Here $\tilde{I} := I \sqcup \cdot$ (see 1.2.1).

equals the sum of I operations $\varphi(\cdot_{M_i}, id_{M_{i'}})_{i' \neq i},\ i \in I$.[115] This amounts to the fact that φ satisfies the Leibnitz rule with respect to the $h(L)$-action on our $\mathcal{D}$-modules. We denote the set of such operations by P^{ch}_{LI}; they define on $\mathcal{M}(X, L)$ a pseudo-tensor structure $\mathcal{M}(X, L)^{ch}$.

(ii) Let R^ℓ be a commutative $\mathcal{D}_X$-algebra, M_i, N are $R^\ell[\mathcal{D}_X]$-modules. Consider the $\mathcal{D}_{X^I}$-algebra $R^{\ell \boxtimes I}$. Then $j^{(I)}_* j^{(I)*} \boxtimes M_i$ and $\Delta^{(I)}_* N$ are $R^{\ell \boxtimes I}$-modules in the obvious way.[116] We say that a chiral operation $\varphi : j^{(I)}_* j^{(I)*} \boxtimes M_i \to \Delta^{(I)}_* N$ is R^ℓ-*polylinear* if it is a morphism of $R^{\ell \boxtimes I}$-modules.

R^ℓ-polylinear chiral operations are closed under composition, so they define a pseudo-tensor structure on $\mathcal{M}(X, R^\ell)$, which we denote by $\mathcal{M}(X, R^\ell)^{ch}$.

§2 Relation to "classical" operations

In this section we show that "classical" operations from Ch 2 can be considered as "symbols", or "classical limits" of chiral operations.

3.2.1. The obvious morphism $\boxtimes L_i \to j^{(I)}_* j^{(I)*} \boxtimes L_i$ yields a map

$$\beta_I \colon P^{ch}_I(\{L_i\}, M) \to P^*_I(\{L_i\}, M). \tag{3.2.1.1}$$

It is clear that β's are compatible with composition of operations, so we have a pseudo-tensor functor

$$\beta \colon \mathcal{M}(X)^{ch} \to \mathcal{M}(X)^* \tag{3.2.1.2}$$

which extends the identity functor on $\mathcal{M}(X)$. It is compatible with the augmentations, so β is an augmented pseudo-tensor functor.

3.2.2. Consider the tensor category $\mathcal{M}(X)^!$ and the pseudo-tensor category $\mathcal{M}(X)^! \otimes \mathcal{L}ie$ (see 1.1.10). So $\mathcal{M}(X)^! \otimes \mathcal{L}ie$ coincides with $\mathcal{M}(X)$ as a usual category, and the corresponding operations are $P^{!\mathcal{L}ie}_I(\{L_i\}, M) := \mathrm{Hom}(\otimes^! L_i, M) \otimes \mathcal{L}ie_I$ where $\mathcal{L}ie$ is the Lie algebras operad.

---

[115] Here $\cdot_{M_i}$, $\cdot_N$ are the $*$ actions of L.

[116] The $R^{\ell \boxtimes I}$-module structure on $\Delta^{(I)}_* N$ comes from the $R^{\ell \otimes I}$-module structure on N (defined by the product morphism of algebras $R^{\ell \otimes I} \to R^\ell$) and the fact that $\Delta^{(I)}_*$ sends $R^{\ell \otimes I} = \Delta^{(I)*} R^{\ell \boxtimes I}$-modules to $R^{\ell \boxtimes I}$-modules (to define the module structure use (2.1.3.1) for $i = \Delta^{(I)}$, $L = R^{\ell \boxtimes I}$, $M = N$).

There is a canonical faithful pseudo-tensor functor

$$\alpha\colon \mathcal{M}(X)^{!}\otimes \mathcal{L}ie \to \mathcal{M}(X)^{ch} \tag{3.2.2.1}$$

which extends the identity functor on $\mathcal{M}(X)$. Namely, the map

$$\alpha_I\colon \mathrm{Hom}(\otimes^{!}L_i, M)\otimes \mathcal{L}ie_I \hookrightarrow P_I^{ch}(\{L_i\}, M) \tag{3.2.2.1}$$

sends $\phi\otimes\nu \in \mathrm{Hom}(\otimes^{!}L_i, M)\otimes \mathcal{L}ie_I$ to a chiral operation equal to composition

$$j_*^{(I)}j^{(I)*}(\boxtimes L_i) = (\boxtimes L_i^{\ell})\otimes j_*^{(I)}j^{(I)*}\omega_X^{\boxtimes I} \to (\boxtimes L_i^{\ell})\otimes \Delta_*^{(I)}\omega_X = \Delta_*^{(I)}(\otimes^{!}L_i) \to \Delta_*^{(I)}M.$$

Here the first arrow is $\kappa(\nu)$ (see 3.1.4) tensored by the identity morphism of $\boxtimes L_i^{\ell}$, the second one is $\Delta_*^{(I)}(\phi)$, and the equalities are standard canonical isomorphisms. We leave it to the reader to check injectivity of α_I (notice that the canonical map $j_*^{(I)}j^{(I)*}\omega_X^{\boxtimes I} \to \Delta_*^{(I)}\omega_X\otimes \mathcal{L}ie_I^{*}$ coming from κ is surjective) and compatibility of α_I's with composition of operations.

3.2.3 Remark. The composition $\beta_I\alpha_I : P_I^{!\mathcal{L}ie} \to P_I^{*}$ vanishes for $|I|\geq 2$. The sequence $0\to P_2^{!\mathcal{L}ie}\to P_2^{ch}\to P_2^{*}$ is exact. Notice that $P_2^{!}\xrightarrow{\sim} P_2^{!\mathcal{L}ie}$, $\phi\mapsto \phi\otimes[\quad]$.

The above pseudo-tensor functors α, β are parts of the following general picture:

3.2.4. Notice that $j_*^{(I)}j^{(I)*}\boxtimes L_i = (j_*^{(I)}j^{(I)*}\omega_X^{\boxtimes I})\otimes(\boxtimes L_i^{\ell})$. So the special filtration $W.$ on $j_*^{(I)}j^{(I)*}\omega_X^{\boxtimes I}$ (see 3.1.5, 3.1.6) yields a finite filtration on $j_*^{(I)}j^{(I)*}\boxtimes L_i$ which we denote also by $W.$. The canonical identifications $gr^W_{-l}j_*^{(I)}j^{(I)*}\omega_X^{\boxtimes I} = \underset{T\in Q(I,l)}{\oplus}\Delta_*^{(I/T)}\omega_X^{\boxtimes T}\otimes \mathcal{L}ie^{*}_{I/T}$ of 3.1.9(i) yields canonical surjections

$$\underset{T\in Q(I,l)}{\oplus}\Delta_*^{(I/T)}\Big(\underset{t\in T}{\boxtimes}\big(\overset{!}{\underset{I_t}{\otimes}}L_i\big)\otimes \mathcal{L}ie^{*}_{I_t}\Big) \twoheadrightarrow gr^W_{-l}j_*^{(I)}j^{(I)*}\boxtimes L_i \tag{3.2.4.1}$$

which are isomorphisms if L_i are $\mathcal{O}_X$-flat.

The above filtration yields a canonical filtration on k-modules of chiral operations:

$$P_I^{ch}(\{L_i\}, M)^n := \mathrm{Hom}(j_*^{(I)}j^{(I)*}\boxtimes L_i/W_{n-1-|I|}j_*^{(I)}j^{(I)*}\boxtimes L_i,\ \Delta_*^{(I)}M). \tag{3.2.4.2}$$

The morphisms (3.2.4.1) yield canonical embeddings

$$gr^n P_I^{ch}(\{L_i\},M) \hookrightarrow \underset{T\in Q(I,|I|-n)}{\oplus} P_T^*(\{\otimes^!_{I_t} L_i\},M)\otimes \mathcal{L}ie_{I/T} \tag{3.2.4.3}$$

which are isomorphisms if L_i are projective $\mathcal{D}_X$-modules.

Notice that the maps α, β from 3.2.1, 3.2.2 occure as "boundary" morphisms of (3.2.4.3). Namely, the morphism (3.2.4.3) for $n=|I|-1$ is always an isomorphism. Its inverse (composed with the embedding $gr^{|I|-1}P_I^{ch} = P_I^{ch\,|I|-1} \hookrightarrow P_I^{ch}$) coincides with α_I of (3.2.2.1). Simlarly, β_I of (3.2.1.1) is the morphism (3.2.4.3) for $n=0$ composed with the projection $P_I^{ch} \twoheadrightarrow P_I^{ch}/P_I^{ch\,1} = gr^0 P_I^{ch}$.

3.2.5. The above filtration is compatible with composition of chiral operations. Namely, for $J \twoheadrightarrow I$ and a J-family of $\mathcal{D}_X$-modules K_j the composition map sends the tensor product $P_I^{ch}(\{L_i\},M)^n \otimes (\otimes_I P_{J_i}^{ch}(\{K_j\},L_i)^{m_i}$ to $P_J^{ch}(\{K_j\},M)^{n+\Sigma m_i}$. Therefore we can define a new pseudo-tensor structure on $\mathcal{M}$ with operations $P^{cl} := gr^{\cdot}P^{ch}$; this is *the classical limit* of chiral structure. We can rewrite (3.2.4.3) as a canonical embedding $P_I^{cl}(\{L_i\},M) \hookrightarrow P_I^c(\{L_i\},M)$ (recall that c operations were defined in 1.4.26 using the compound pseudo-tensor structure $\mathcal{M}^{*!}$) which is isomorphism if L_i are projective $\mathcal{D}_X$-modules. It follows from definitions that this embedding is compatible with composition of operations. So we defined a canonical fully faithful embedding of pseudo-tensor categories

$$\mathcal{M}(X)^{cl} \hookrightarrow \mathcal{M}(X)^c \tag{3.2.5.1}$$

which extends the identity functor on $\mathcal{M}$.

§3 Chiral algebras and modules

3.3.1. For a k-operad $\mathcal{B}$ a $\mathcal{B}^{ch}$ *algebra* on X is a $\mathcal{B}$ algebra in the pseudo-tensor category $\mathcal{M}(X)^{ch}$. The category $\mathcal{B}(\mathcal{M}(X)^{ch})$ of $\mathcal{B}^{ch}$ algebras will also be denoted by $\mathcal{B}^{ch}(X)$. The canonical pseudo-tensor functors α, β from §3.2 yield the functors between the categories of $\mathcal{B}$ algebras

$$\mathcal{B}(\mathcal{M}(X)^!\otimes \mathcal{L}ie) \xrightarrow{\alpha^{\mathcal{B}}} \mathcal{B}^{ch}(X) \xrightarrow{\beta^{\mathcal{B}}} \mathcal{B}^*(X) \tag{3.3.1.1}$$

We will be interested in Lie^{ch} algebras. According to 1.1.10 one has $\mathcal{L}ie(\mathcal{M}(X)^! \otimes \mathcal{L}ie) = \mathcal{C}om^!(X) := \mathcal{C}om(\mathcal{M}(X)^!)$, so we may rewrite (3.3.1.1) as

$$\mathcal{C}om^!(X) \xrightarrow{\alpha^{\mathcal{L}ie}} \mathcal{L}ie^{ch}(X) \xrightarrow{\beta^{\mathcal{L}ie}} \mathcal{L}ie^*(X). \tag{3.3.1.2}$$

Note that $\beta^{\mathcal{L}ie}$ is a faithful functor, and $\alpha^{\mathcal{L}ie}$ is a fully faithful embedding (since α is a faithful pseudo-tensor functor).

3.3.2. For a Lie^{ch} algebra A we denote by $\mu = \mu_A \in P_2^{ch}(\{A, A\}, A)$ the commutator and by $[\ ] = [\ ]_A \in P_2^*(\{A, A\}, A)$ the $*$ bracket $\beta(\mu)$. We usually call μ the *chiral product*. Set $A^{(*)} := \beta^{\mathcal{L}ie}(A) \in \mathcal{L}ie(\mathcal{M}^*)$, i.e., $A^{(*)}$ is A considered as a Lie* algebra with bracket $[\ ]_A$.

Consider the sheaf of Lie algebras $h(A) = h(A^{(*)})$ (see 2.5.3). It acts on the $\mathcal{D}$-module A (the adjoint action) by derivations of μ_A (see the end of 1.2.18).

A Lie^{ch} algebra A is said to be *commutative* if $[\ ]_A = 0$. Denote by $\mathcal{L}ie^{ch}_{com}(X) \subset \mathcal{L}ie^{ch}(X)$ the full subcategory of commutative Lie^{ch} algebras. It follows from 3.2.3 that $\alpha^{\mathcal{L}ie}(\mathcal{C}om^!(X)) = \mathcal{L}ie^{ch}_{com}(X)$, i.e., we have the equivalence of categories

$$\alpha^{\mathcal{L}ie}\colon \mathcal{C}om^!(X) \xrightarrow{\sim} \mathcal{L}ie^{ch}_{com}(X). \tag{3.3.2.1}$$

3.3.3. Let A be a Lie^{ch} algebra. A *unit in* A is a morphism of $\mathcal{D}$-modules $1 = 1_A\colon \omega_X \to A$ (i.e., 1 is a horizontal section of A^ℓ) such that $\mu_A(1, id_A) \in P_2^{ch}(\{\omega_X, A\}, A)$ is the unit operation ε_A (see 3.1.11). A unit in A is unique if it exists.

A Lie^{ch} algebra with unit is called *unital Lie^{ch} algebra* or simply *chiral algebra.*[117] Denote by $\mathcal{CA}(X) \subset \mathcal{L}ie^{ch}(X)$ the subcategory of chiral algebras and morphisms preserving units. One has an obvious "adding of unit" functor[118] $\mathcal{L}ie^{ch}(X) \to \mathcal{CA}(X), \quad B \mapsto B \oplus \omega_X$, left adjoint to the embedding $\mathcal{CA}(X) \hookrightarrow \mathcal{L}ie^{ch}(X)$.

Chiral algebras form a sheaf of categories $\mathcal{CA}(X_{\text{ét}})$ on the étale topology of X.

Let L be a Lie* algebra. An *L-action on A* is a Lie* action of L on the $\mathcal{D}$-module A such that μ_A is L-compatible (see 3.1.13(i)).[119] E.g., we have the adjoint $A^{(*)}$-action on A.

[117] We will also refer to Lie^{ch} algebras as chiral algebras without unit.

[118] The Jacobi identity for $B \oplus \omega_X$ follows from 3.1.12(ii).

[119] The unit is automatically fixed by any L-action (as well as any horizontal section of A^ℓ).

Let $\mathcal{CA}(X)_{com} \subset \mathcal{CA}(X)$ be the full subcategory of commutative chiral algebras. The above equivalence (3.3.2.1) identifies the corresponding subcategories of unital algebras, so we have a canonical equivalence (see 2.3.1 for notation)

$$\mathcal{C}omu_{\mathcal{D}}(X) = \mathcal{C}omu^!(X) \xrightarrow{\sim} \mathcal{CA}(X)_{com}. \tag{3.3.3.1}$$

Every chiral algebra A admits the maximal commutative quotient.[120]

3.3.4. For a chiral algebra A an *A-module* (or *chiral A-module*) always means *unital* A-module, i.e., this is an A-module M (here A is considered as a Lie^{ch} algebra) such that for the A-action $\mu_M \in P_2^{ch}(\{A, M\}, M)$ the operation $\mu_M(1_A, id_M) \in P_2^{ch}(\{\omega_X, M\}, M)$ is the unit operation ε_M (see 3.1.11). We denote the category of A-modules by $\mathcal{M}(X, A)$. According to 1.2.18, $\mathcal{M}(X, A)$ carries an abelian augmented pseudo-tensor structure; we denote it $\mathcal{M}(X, A)^{ch}$. For $M_i, N \in \mathcal{M}(X, A)$ the corresponding vector space of *chiral A-operations* is denoted by $P_{AI}^{ch}(\{M_i\}, N) \subset P_I^{ch}(\{M_i\}, N)$. For any A-module M the Lie algebra $h(A)$ acts canonically on M (considered as a plain $\mathcal{D}$-module); every operation from P_{AI}^{ch} satisfies Leibnitz rule with respect to this action. The augmentation functor on $\mathcal{M}(X, A)^{ch}$ is $M \mapsto h^A(M) := h(M)^{h(A)}$.

Example. For the unit chiral algebra ω_X the forgetful functor $\mathcal{M}(X, \omega_X)^{ch} \to \mathcal{M}(X)^{ch}$ is an equivalence of augmented pseudo-tensor categories (see 3.1.12(ii)).

Remarks. (i) If N is a left $\mathcal{D}_X$-module, M a chiral A-module then $N \otimes M$ is a chiral A-module. So $\mathcal{M}(X, A)^{ch}$ is a Module over the tensor category $\mathcal{M}^{\ell}(X)$.

(ii) A-modules localize in the obvious manner, so we have a sheaf of augmented pseudo-tensor categories $\mathcal{M}(X_{\text{ét}}, A)^{ch}$.

(iii) Let $a : P \hookrightarrow A$ be a sub$\mathcal{D}$-module which generates A as a chiral algebra. Then for M_i, N as above an operation $\varphi \in P_I^{ch}(\{M_i\}, N)$ is a chiral A-operation if and only if the operations $\mu_N(a, \varphi), \sum_{i \in I} \varphi(id_{M_{j \neq i}}, \mu_{M_i}(a, id_{M_i})) \in P_{\tilde{I}}^{ch}(\{P, M_i\}, N)$ coincide.

Any A-module is automatically (via the functor β) an $A^{(*)}$-module. Precisely, we have

[120] which is the quotient of A modulo the ideal generated by the image of $[\ ]_A$.

the pseudo-tensor functor

$$\beta_A\colon \mathcal{M}(X,A)^{ch} \to \mathcal{M}(X,A^{(*)})^*. \tag{3.3.4.1}$$

Remark. For $M_i, N \in \mathcal{M}(X,A)$ one has $P^{ch}_{AI}(\{M_i\},N) \subset P^{ch}_{A^{(*)}I}(\{M_i\},N)$ (see 3.1.13(i) for notation), so (3.3.4.1) is the composition of pseudo-tensor functors

$$\mathcal{M}(X,A)^{ch} \to \mathcal{M}(X,A^{(*)})^{ch} \to \mathcal{M}(X,A^{(*)})^*. \tag{3.3.4.2}$$

3.3.5 Remark. An A-module structure on M amounts to a structure of chiral algebra on $A \oplus M$ such that $A \hookrightarrow A \oplus M$ and $A \oplus M \twoheadrightarrow A$ are morphisms of chiral algebras and the restriction of the chiral product to $M \times M$ vanishes.

3.3.6. Let M be an A-module and $\gamma \in \Gamma(X,M)$ a section.

We say that $a \in A$ *kills* γ if $\mu_M(fa \boxtimes \gamma) \in \Delta_* M$ vanishes for every $f \in j_* j^* \mathcal{O}_{X\times X}$. Such a form a subsheaf $\mathcal{A}nn(\gamma) \subset A$ called *the annihilator of* γ. This is an ideal in A.

The *centralizer* $\mathcal{C}ent(\gamma)$ of γ consists of sections $a \in A$ that commute with γ, i.e., $\mu_M(a \boxtimes \gamma) = 0$. This is a chiral subalgebra of A. We say that γ is an A-*central* section of M if $\mathcal{C}ent(\gamma) = A$.

The annihilator and centralizer of a section are compatible with étale localization of X.

Exercise. Show that the map $\varphi \mapsto \varphi(1)$ identifies $\mathrm{Hom}_{\mathcal{M}(X,A)}(A,M)$ with the vector space of all horizontal A-central sections of M^ℓ.

The *annihilator* $\mathcal{A}nn(M) = \mathcal{A}nn(A,M)$ *of* M is the subsheaf of A whose sections over an affine open U are those a that kill every section γ of M over U. Similarly, the *centralizer* $\mathcal{C}ent(M) = \mathcal{C}ent(A,M)$ *of* M is the subsheaf of A whose sections over U as above are those a that commute with every γ. Both these subsheaves are $\mathcal{O}_X$-submodules of A. If we know that $\mathcal{A}nn(M)$ (resp. $\mathcal{C}ent(M)$) is a quasicoherent $\mathcal{O}_X$-module then it is an ideal (resp. a chiral subalgebra) in A. This always happens if M is $\mathcal{O}_X$-flat.

We say that M is *commutative* (or the A-action on M is commutative) if $\mathcal{C}ent(M) = A$, or, equivalently, if every $m \in M$ is A-central.

The centralizer of A (considered as an A-module) is called the *center of* A; we denote it by $Z(A)$. If $Z(A)$ is $\mathcal{O}_X$-quasicoherent (see above) then this is a commutative subalgebra of A.

We denote by $\mathcal{M}(X,A)^{ch}_{com} \subset \mathcal{M}(X,A)^{ch}$ the full pseudo-tensor subcategory of commutative modules. The subcategory $\mathcal{M}(X,A)_{com} \subset \mathcal{M}(X,A)$ is closed under direct sums and subquotients. Any A-module M has maximal commutative sub- and quotient modules which we denote, respectively, by M^{com} and M_{com}. Notice that M_{com} is the quotient of M modulo the A-submodule generated by the image of the $*$ action of $A^{(*)}$ on M. So the functor $M \mapsto M_{com}$ is compatible with the étale localization.

Every chiral algebra A admits the maximal commutative quotient algbera A_{com}. As an A-module it equals the maximal commutative A-module quotient of A.

3.3.7 Lemma. The obvious functor $\mathcal{M}(X,A_{com})^{ch}_{com} \to \mathcal{M}(X,A)^{ch}_{com}$ is an equivalence of pseudo-tensor categories.

Proof. An equivalent statement: the action of A on any commutative A-module factors through A_{com}. The proof is left to the reader. □

3.3.8. Let R be a commutative chiral algebra, so R^ℓ is a commutative $\mathcal{D}_X$-algebra. The functor α identifies R^ℓ-actions on a $\mathcal{D}_X$-module M with *commutative* chiral R-actions. We get a canonical equivalence of categories

$$\mathcal{M}(X,R^\ell) \xrightarrow{\sim} \mathcal{M}(X,R)_{com}. \tag{3.3.8.1}$$

We leave it to the reader to check that a chiral operation φ between $R^\ell[\mathcal{D}_X]$-modules is R^ℓ-polylinear (see 3.1.13(ii)) if and only if it satisfies Leibnitz rule with respect to the chiral actions of R on these modules.[121] So we have a canonical equivalence of pseudo-tensor categories

$$\mathcal{M}(X,R^\ell)^{ch} \xrightarrow{\sim} \mathcal{M}(X,R)^{ch}_{com}. \tag{3.3.8.2}$$

3.3.9. Everything said in this section remains valid if we replace $\mathcal{M}(X)^{ch}$ by $\mathcal{M}(X,R^\ell)^{ch}$. We leave it to the reader to check that (3.3.8.2) identifies chiral algebras in $\mathcal{M}(X,R^\ell)$ with pairs (A,i) where A is a chiral algebra on X and $i : R \to A$ is a morphism of chiral algebras such that A is a commutative R-module; we call such pair (A,i) a *chiral R^ℓ-algebra* or a

[121] For a generalization of this statement to the case of arbitrary chiral algebras see 3.3.19.

chiral algebra on SpecR^ℓ. An A-module in $\mathcal{M}(X, R^\ell)^{ch}$ is the same as an A-module M such that the R-action on M (via the structure morphism i) is commutative; such objects are called *R-commutative A-modules* or *A-modules on* Spec$R$.

Recall that $R^\ell[\mathcal{D}_X]$-modules have local nature with respect to étale topology of SpecR^ℓ (see 2.3.6). R^ℓ-polylinear chiral operations also have local nature. So for every algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ we have the pseudo-tensor category $\mathcal{M}(\mathcal{Y})^{ch}$ of $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules (see 2.3.6), the category $\mathcal{CA}(\mathcal{Y})$ of chiral algebras on $\mathcal{Y}$, and for $\mathcal{A} \in \mathcal{CA}(\mathcal{Y})$ the pseudo-tensor category $\mathcal{M}(\mathcal{Y}, \mathcal{A})^{ch}$ of $\mathcal{A}$-modules on $\mathcal{Y}$, as well as the corresponding sheaves of categories on $\mathcal{Y}_{\text{ét}}$.

Example. If $\mathcal{Y}$ is constant, i.e., $\mathcal{Y} = X \times Y$ with a "trivial" connection, then $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules are the same as Y-families of $\mathcal{D}$-modules on X. Similarly, chiral algebras on $\mathcal{Y}$ are the same as Y-families of chiral algebras on X, etc.

3.3.10. Let us explain why coisson algebras are "classical limits" of chiral algebras.

Let A_t be a one-parameter flat family of chiral algebras, i.e., A_t is a chiral $k[t]$-algebra which is flat as a $k[t]$-module (see 3.3.9). Assume that $A_{t=0} := A/tA$ is a commutative chiral algebra. This means that the $*$ bracket $[\quad]_t$ on A_t is divisible by t. Thus $\{\quad\}_t := t^{-1}[\quad]$ is a Lie* bracket on A_t; the corresponding Lie* algebra acts on the chiral algebra A_t (see 3.3.3) by adjoint action. Reducing this picture modulo t we see[122] that $A^\ell_{t=0}$ is a commutative $\mathcal{D}_X$-algebra, and $\{\quad\} := \{\quad\}_{t=0}$ is a coisson bracket on it.

Remark. The above construction is also a direct consequence of 3.2.5 and 1.4.27.

3.3.11. Let A be a Lie^{ch} algebra. A *filtration* on A is an increasing sequence of $\mathcal{D}$-submodules $A_0 \subset A_1 \subset .. \subset A$ such that $\cup A_i = A$ and $\mu(j_*j^*A_i \boxtimes A_j) \subset \Delta_* A_{i+j}$. Then gr$A$ is a Lie^{ch} algebra in the obvious way.

If A is a chiral algebra then, unless it is stated explicitely otherwise, we assume that our filtration is *unital*, i.e., $1_A \in A_0$. Then gr.A is a chiral algebra. The filtration is *commutative* if gr.A is a commutative chiral algebra. Then gr.A^ℓ is a coisson algebra: the coisson bracket $\{\quad\} \in P_2^*(\{\mathrm{gr}_i A, \mathrm{gr}_j A\}, A_{i+j-1})$ comes from the Lie* bracket on A.

[122] Look in 3.3.3.

Set $A_t := \oplus A_i$. This is a chiral $k[t]$-algebra in the obvious way[123] which is $k[t]$-flat. One has $A_{t=1} = A$, $A_{t=0} = \mathrm{gr}A$. If our filtration is commutative then we are in the situation of 3.3.10, and the above coisson bracket equals the one from 3.3.10.

§4 Local associative algebras

We assign to every chiral algebra A certain sheaf of topological associative algebras A^{as} equipped with a connection which governs the category of A-modules (and uniquely determines A itself).

3.4.1. Let A be a chiral algebra on X. Now, as promised in the beginning of this section, we are going to construct a sheaf of topological associative algebra A^{as} on X equipped with a connection ∇.[124] It yields a description of the category of A-modules similar to the one given in 3.3.8 in the commutative case.

Here is a short definition of A^{as}. Consider the forgetting of the A-action functor which assigns to every A-module M the same M considered as a plain $\mathcal{O}_Y$-module. Our A^{as} is the sheaf of endomorphisms of this functor. Unfortunately this definition says little about the structure of A^{as}. So we will give another "concrete" definition of A^{as} returning to the previous one later.

3.4.2. Let Y be a scheme[125] which we assume to be quasicompact and quasiseparated,[126] $x \in X(Y)$, $i_x : Y \hookrightarrow X \times Y$ be the graph of x, and $j_x : U_x \hookrightarrow X \times Y$ its complement. For a chiral algebra A consider the chiral algebras $A \boxtimes \mathcal{O}_Y$ and $j_{x*}j_x^*(\mathcal{A} \boxtimes \mathcal{O}_Y)$ on $X \times Y$ (see Example above). Denote by Ξ_x^{as} the set of chiral subalgebras $A_\xi \subset j_{x*}j_x^*(\mathcal{A} \boxtimes \mathcal{O}_Y)$ which coincide with $A \boxtimes \mathcal{O}_Y$ over U_x. Notice that if $A_\xi, A_{\xi'}$ are in Ξ_x^{as} then such is $A_\xi \cap A_{\xi'}$, so Ξ_x^{as} is a topology on $j_{x*}j_x^*(\mathcal{A} \boxtimes \mathcal{O}_Y)$ at x (we use terminology from 2.1.13 in version of 2.1.15).

---

[123] The chiral product on A_t is compatible with the grading and comes from the chiral product on A; multiplication by t is the sum of embeddings $A_i \hookrightarrow A_{i+1}$.

[124] A physicist would probably call A^{as} "the sheaf of observables".

[125] The reader can assume that Y is a point; the general case brings purely notational complications.

[126] The assumption is needed to assure that topological $\mathcal{O}_Y$-modules have local nature, i.e., proobjects in the category of quasicoherent $\mathcal{O}$-modules form a sheaf of categories on $Y_{\mathcal{Z}ar}$, and our constructions are compatible with localization of Y.

The fibers $A^{\ell}_{\xi x} := i_x^* A^{\ell}_{\xi} = i_x^!(j_{x*}j_x^*(\mathcal{A} \boxtimes \mathcal{O}_Y)/A_\xi)$ form a Ξ^{as}_x-projective system of $\mathcal{O}_Y$-modules connected by surjective morphisms. We denote by A^{as}_x its projective limit. Equivalently (see 2.1.15), A^{as}_x is the completion of $i_x^! h(j_{x*}j_x^*(\mathcal{A} \boxtimes \mathcal{O}_Y))$ with respect to the Ξ^{as}_x-topology. So every $A^{\ell}_{\xi x}$ is a quotient of A^{as}_x modulo an open submodule I_ξ.

We leave it to the reader to check (cf. 2.1.15(i)) that the formation A^{as}_{xY} is compatible with the base change (i.e., for every $f : Y' \to Y$ the pro-$\mathcal{O}_{Y'}$-modules $f^* A^{as}_{xY}$ and $A^{as}_{xfY'}$ coincide). In the universal case $Y = X$, $x = id_X$ (so $i_x = \Delta$, $U_x = U$) the corresponding topological $\mathcal{O}_X$-module is denoted simply by A^{as}.

Any automorphism τ of Y such that $id_X \times \tau$ preserves U_x acts on our picture, hence acts on the A^{as}_x. Since our construction is compatible with base change we see that any infinitesimal automorphism of Y acts canonically on A^{as}_x. Therefore if Y is a smooth variety then A^{as}_x carries a canonical flat connection (i.e., it is a topological $\mathcal{D}_Y$-module). In particular the topological $\mathcal{O}_X$-module A^{as} carries a canonical connection ∇.

We are going to define on A^{as}_x a canonical structure of associative algebra such that I_ξ become right ideals. To do this we need an auxiliary lemma. Let B be any chiral algebra on $X \times Y$. Denote by $\mathcal{M}(X \times Y, B)_x$ the category of B-modules on $X \times Y$ supported set-theoretically at (the graph of) x. Let $1_x \in i_x^* B^{\ell} = i_x^!(j_{x*}j_x^* B/B)$ be the value at x of the unit section of B^{ℓ}. We may consider 1_x as a section of $j_{x*}j_x^* B/B$ supported scheme-theoretically at x; it is obviously B-central.

3.4.3 Lemma. (i) The obvious functor $\mathcal{M}(X \times Y, j_{x*}j_x^* B)_x \to \mathcal{M}(X \times Y, B)_x$ is an equivalence of categories.

(ii)[127] For every $M \in \mathcal{M}(X \times Y, B)$ the map $\varphi \mapsto \varphi(1_x)$ identifies $\mathrm{Hom}(j_{x*}j_x^* B/B, M)$ with the space of B-central sections of M supported scheme-theoretically at x.

Proof. (i) Let M be a $\mathcal{D}_{X \times Y/Y}$-module supported at the graph of x. We want to show that every B-action on M extends uniquely to a $j_{x*}j_x^* B$-action, and this extension is compatible with morphisms of M's. Notice that for every finite family of $\mathcal{D}_{X \times Y/Y}$-modules

[127] cf. Exercise in 3.3.6.

B_i, $i \in I$, the obvious morphism of $\mathcal{D}_{X^{\tilde{I}} \times Y}$-modules[128]

$$j_*^{(\tilde{I})} j^{(\tilde{I})*}((\underset{I}{\boxtimes} B_i) \boxtimes M) \to j_*^{(\tilde{I})} j^{(\tilde{I})*}((\underset{I}{\boxtimes} j_{x*} j_x^* B_i) \boxtimes M)$$

is an isomorphism. Thus $P_{\tilde{I}}^{ch}(\{j_{x*}j_x^*B_i, M\}, M) \xrightarrow{\sim} P_{\tilde{I}}^{ch}(\{B_i, M\}, M)$. This implies the desired statement (take $B_i = B$).

(ii) Since 1_x generates $j_{x*}j_x^*B/B$ our map is injective. The inverse map is constructed as follows. Assume we have a section m of M as in (ii); let us define the corresponding morphism ψ_m. We may assume that M is generated by m, hence M is supported at x. Thus, by (i), M is a $j_{x*}j_x^*B$-module. Consider the chiral action morphism $j_{x*}j_x^*B \boxtimes M \to \Delta_* M$. Pulling it back by $(id_X \times i_x)^!$ we get a morphism of $\mathcal{D}_{X \times Y/Y}$-modules $\psi : j_{x*}j_x^*B \underset{\mathcal{O}_Y}{\otimes} i_x^! M \to M$. Since m is B-central the morphism $j_{x*}j_x^*B \to M$, $b \mapsto \psi(b \otimes m)$, vanishes on B, so we have $\psi_m : j_{x*}j_x^*B/B \to M$. This is the morphism we want. □

3.4.4. Applying 3.4.3(i) to our situation we see that for a $\mathcal{D}_{X \times Y/Y}$-module supported at x an A_ξ-action on it amounts to a $j_{x*}j_x^*(\mathcal{A} \boxtimes \mathcal{O}_Y)$-action. Thus the categories of A_ξ-modules supported at x are the same for all $A_\xi \in \Xi_x^{as}$, so we denote them simply $\mathcal{M}(X \times Y, A)_x$.

Now for every $\xi \in \Xi_x^{as}$ we have the A_ξ-module $\Phi_\xi := j_{x*}j_x^*(\mathcal{A} \boxtimes \mathcal{O}_Y)/A_\xi$ supported at x. For $A_{\xi'} \subset A_\xi$ the canonical projection $\Phi_{\xi'} \twoheadrightarrow \Phi_\xi$ is a morphism of $A_{\xi'}$-modules. So, according to 3.4.3(i), we have a Ξ_x^{as}-projective system $\Phi = \{\Phi_\xi\}$ in $\mathcal{M}(X \times Y, A)_x$.

One has $i_x^! \Phi = A_x^{as}$. The elements $1_x \in A_{\xi x}^{\ell}$ form a section 1_x of A_x^{as}.

Take any $M \in \mathcal{M}(X \times Y, A)_x$. For every section γ of $i^! M$ its centralizer (see 3.3.6; here we consider γ as a section of M) belongs to Ξ_x^{as}. Therefore, by 3.4.3(ii), we have

$$\mathrm{Hom}(\Phi, M) \xrightarrow{\sim} \Gamma(Y, i_x^! M), \quad \psi \mapsto \psi(1_x). \tag{3.4.4.1}$$

In particular, $\Gamma(Y, A_x^{as}) = \mathrm{End}\Phi$. We define on $\Gamma(Y, A_x^{as})$ the structure of an associative algebra so that this identification becomes an antiisomorphism of algebras (so Φ is a right $\Gamma(Y, A_x^{as})$-module). The section 1_x is the unit in $\Gamma(Y, A_x^{as})$. We see that every $\Gamma(Y, i_x^! M)$ is a left unital $\Gamma(Y, A_x^{as})$-module. By construction the product and the action

[128] Here $\tilde{I} := I \sqcup \cdot$.

are continuous.[129] Localizing this picture with respect to Y we see that A_x^{as} is a sheaf of associative topological $\mathcal{O}_Y$-algebras and $i_x^!M$ is a discrete left A_x^{as}-module.

Exercise. Recall that the Lie* bracket on A yields a Lie bracket on $h_x(j_{x*}j_x^*A \boxtimes \mathcal{O}_Y) = i_x^!h(j_{x*}j_x^*A \boxtimes \mathcal{O}_Y)$. Show that the obvious morphism[130] with dense image

$$h_x(j_{x*}j_x^*A \boxtimes \mathcal{O}_Y) \to A_x^{as} \tag{3.4.4.2}$$

is a morphism of Lie $\mathcal{O}_Y$-algebras.

Remark. Assume that Y is a point, i.e., x is a closed point of X and $j_x : U_x \hookrightarrow X$ is its complement. Then $A_x^{as} = (j_{x*}j_x^*A)_x^{as}$, so our topological associative algebra depends only on the restriction of A to U_x. If B is a chiral algebra on U_x we set $B_x^{as} := (j_{x*}B)_x^{as}$.

Denote by A_x^{as}mod the category of discrete left unital A_x^{as}-modules quasicoherent as $\mathcal{O}_Y$-modules.

3.4.5 Lemma. The functor $i_x^! : \mathcal{M}(X \times Y, A)_x \to A_x^{as}$mod is an equivalence of categories.

Proof. Let us define the inverse functor. Recall that by Kashiwara's lemma (see 2.1.3, 2.1.15) the functor $i_x^!$ is an equivalence between the category of right $\mathcal{D}_{X\times Y/Y}$-modules supported at x and that of quasicoherent $\mathcal{O}_Y$-modules. The inverse functor sends an $\mathcal{O}_Y$-module V to $i_{x*}V := (j_{x*}j_x^*\omega_X \boxtimes V)/\omega_X \boxtimes V$.

Now assume that V is a discrete A_x^{as}-module. Then $i_{x*}V$ is a chiral $A \boxtimes \mathcal{O}_Y$-module. Indeed, by Kashiwara's lemma, one has $i_{x*}A_x^{as} = \Phi$. So, since i_{x*} commutes with direct limits, we have[131] $i_{x*}V = \Phi \underset{A_x^{as}}{\otimes} p_Y^!V$ which yields the promised $A \boxtimes \mathcal{O}_Y$-module structure. The functor $i_{x*} : A_x^{as}\text{mod} \to \mathcal{M}(X \times Y, A)_x$ is obviously inverse to $i_x^!$. □

3.4.6. Let us consider the universal case of the topological associative algebra A^{as} on X (so we have $Y = X$ and $i_x = \Delta$, see 3.4.2). The canonical connection ∇ on A^{as} is obviously compatible with the product.

[129] With respect to the topology on A_x^{as} defined in 3.4.2 and the discrete topology on $i_x^!M$.
[130] It comes from the maps $j_{x*}j_x^*A \boxtimes \mathcal{O}_Y \to \Phi_\xi$ together with identifications $i_x^!\Phi_\xi = h_x\Phi_\xi$.
[131] Here $p_Y : X \times Y \to Y$ is the projection.

Let M be an A-module. Let us define on M^ℓ a canonical A^{as}-action compatible with connections. Consider the $A \boxtimes \mathcal{O}_X$-module $M \boxtimes \mathcal{O}_X$ equipped with the obvious connection along the second variable. The module $(j_* j^* M \boxtimes \mathcal{O}_X)/M \boxtimes \mathcal{O}_X$ supported at the diagonal carries the same structure. Therefore $i^!_x((j_* j^* M \boxtimes \mathcal{O}_X)/M \boxtimes \mathcal{O}_X)$ is an A^{as}-module equipped with a connection, and the A^{as}-action is obviously compatible with connections. This module equals M^ℓ as a plain left $\mathcal{D}_X$-module, and we are done.

Our canonical connection ∇ defines on $A^{as}[\mathcal{D}_X] := A^{as} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ the structure of an associative algebra.[132] A *discrete* $A^{as}[\mathcal{D}_X]$*-module* is a left unital $A^{as}[\mathcal{D}_X]$-module which is discrete as an A^{as}-module and quasicoherent as an $\mathcal{O}_X$-module. Equivalently, this is a left discrete A^{as}-module, quasicoherent as an $\mathcal{O}_X$-module, equipped with a connection such that the A^{as}-action morphism is horizontal, or a $\mathcal{D}_X$-module equipped with a left discrete horizontal A^{as}-action. We denote the category of discrete $A^{as}[\mathcal{D}_X]$-modules by $\mathcal{M}^\ell(X, A^{as})$.

3.4.7 Proposition. The functor $\mathcal{M}(X, A) \to \mathcal{M}^\ell(X, A^{as})$, $M \mapsto M^\ell$, is an equivalence of categories.

Proof. The equivalence from 3.4.5 identifies discrete $A^{as}[\mathcal{D}_X]$-modules with diagonal supported chiral $A \boxtimes \mathcal{O}_X$-modules equipped with a connection along the second variable compatible with the chiral action. By Kashiwara's lemma these are the same as A-modules on X. The resulting equivalence is inverse to the one of the proposition. □

Now it is clear that our A^{as} coincides with A^{as} defined back in 3.4.1.

3.4.8. For a $\mathcal{D}_X$-module M the above 1-1 correspondence between horizontal discrete A^{as}-actions and chiral A-actions on M can be written explicitly as follows.

The non-quasicoherent $\mathcal{O}_X$-module $A^{as} \otimes M$ carries a topology whose opens are subsheaves V that satisfy the following property: for every local section $m \in M$ there exist an open ideal $I_\xi \subset A^{as}$ such that $I_\xi \otimes m \subset V$. It is clear that any open contains a smaller open V such that it is an $\mathcal{O}$-submodule and $A^{as} \otimes M/V$ is a quasicoherent $\mathcal{O}_X$-module.

[132] It is characterized by properties that the maps $A^{as}, \mathcal{D}_X \to A^{as}[\mathcal{D}_X]$, $a \mapsto a \otimes 1, \partial \mapsto 1 \otimes \partial$, are morphisms of algebras, and for $a \in A^{as}, \tau \in \Theta_X$ one has $\tau a - a\tau = \tau(a)$.

Denote the completion by $A^{as}\hat{\otimes}M$. The $\mathcal{D}_X$-action on $A^{as}\otimes M$ extends by continuity to our completion.

The sheaf $j_*j^*A\boxtimes M$ carries a topology whose opens are subsheaves W that satisfy the following property: for every local section $m\in M$ there exists $A_\xi\subset j_*j^*(A\boxtimes\mathcal{O}_X)$ in Ξ^{as}_Δ such that $A_\xi\otimes m\subset W$. Every open contains a smaller open W such that W is a quasicoherent $\mathcal{D}_X\boxtimes\mathcal{O}_X$-module. The completion $(j_*j^*A\boxtimes M)^\wedge$ is a topological $\mathcal{D}_{X\times X}$-module.

The identifications of $\mathcal{D}_X\boxtimes\mathcal{O}_X$-modules $j_*j^*A\boxtimes\mathcal{O}_X/A_\xi=\Delta_*A^\ell_{\xi\Delta}=\Delta_*(A^{as}/I_\xi)$ (see 3.4.2) yield a canonical isomorphism of topological $\mathcal{D}_{X\times X}$-modules

$$(j_*j^*A\boxtimes M)^\wedge\xrightarrow{\sim}\Delta_*(A^{as}\hat{\otimes}M). \tag{3.4.8.1}$$

Now every chiral A-action on M is automatically continuous in the above topology, so it can be considered as a morphism $(j_*j^*A\boxtimes M)^\wedge\to\Delta_*M$. Similarly, every discrete A^{as}-action on M can be cosidered as a morphism $A^{as}\hat{\otimes}M\to M$. A chiral A-action μ_M is identified with an A^{as}-action $\cdot_M$ if (3.4.8.1) identifies $\Delta_*(\cdot_M)$ with μ_M.

3.4.9. Let us lift the equivalence of 3.4.7 to the pseudo-tensor level. Let M^ℓ_i,N^ℓ be discrete $A^{as}[\mathcal{D}_X]$-modules. A chiral operation $\varphi: j^{(I)}_*j^{(I)*}\boxtimes M_i\to\Delta^{(I)}_*N$ is A^{as}-*polylinear* if it satisfies the following condition. For $i\in I$ consider the (non-quasicoherent) associative $\mathcal{D}_{X^I}$-algebra $p_i^*A^{as}$ (here $p_i:X^I\to X$ is the projection). Then $j^{(I)}_*j^{(I)*}\boxtimes M_i$ and $\Delta^{(I)}_*N$ are $p_i^*A^{as}$-modules in the obvious way.[133] We demand that φ is a morphism of $p_i^*A^{as}$-modules for every $i\in I$.

A^{as}-polylinear chiral operations are stable with respect to compositions, so discrete $A^{as}[\mathcal{D}_X]$-modules form a pseudo-tensor category $\mathcal{M}^\ell(X,A^{as})^{ch}$.

Remark.[134] In fact, by Remark in 3.5.2, $\Delta^{(I)}_*N$ is a $\hat{\Delta}^{(I)}_*A^{as}$-module, and the above $p_i^*A^{as}$-action on it comes from the obvious morphism $p_i^*A^{as}\to\hat{\Delta}^{(I)}_*A^{as}$ (see 3.5.5).

[133] The $p_i^*A^{as}$-action on $\Delta^{(I)}_*N$ comes from the A^{as}-action on N via (2.1.3.1) (for $i=\Delta^{(I)}$, $L=A^{as}$, $M=N$).

[134] We use notation from 3.5.2, 3.5.5.

3.4.10 Lemma. A chiral operation φ is A^{as}-polylinear if and only if it satisfies the Leibnitz property with respect to the chiral A-actions on M_i, N. So we have the equivalence of pseudo-tensor categories $\mathcal{M}(X,A)^{ch} \xrightarrow{\sim} \mathcal{M}^{\ell}(X,A^{as})^{ch}$.

Proof. For $i \in I$ let $\Delta^i : X^I \to X \times X^I$ be the diagonal $x = x_i$. Denote by $\tilde{A}$ the chiral algebra $A \boxtimes \mathcal{O}_{X^I}$ localized at all the diagonals Δ^i, $i \in I$. Let Ξ^i be the topology on $\tilde{A}$ formed by chiral subalgebras A^i_ξ that coincide with $\tilde{A}$ outside of the diagonal Δ^i and such that the multiplication on A^i_ξ by an equation of every diagonal $\Delta^{i'}$, $i' \neq i$, is invertible.

The topologies Ξ^i on $\tilde{A}$ are independent, i.e., every I-tuple $\{A^i_\xi\}_{i\in I}$ is self-transversal: for every i one has $A^i_\xi + \bigcap\limits_{i'\neq i} A^{i'}_\xi = \tilde{A}$.

For a section $m \in j^{(I)}_* j^{(I)*} \boxtimes M_i$ consider the morphism[135] $\mu^i_m : \tilde{A} \to \Delta^i_* j^{(I)}_* j^{(I)*} \boxtimes M_i$ of the chiral A-action on m along the ith variable. It is continuous with respect to the Ξ^i-topology, i.e., μ^i_m vanishes on some $A^i_\xi(m) \in \Xi^i$.

Our φ satisfies the Leibnitz rule if for every m the sum of morphisms μ^i_m equals to the chiral action of A on $\varphi(m)$.

By transversality it suffices to check this condition on every $B_i(m) := \bigcap\limits_{i'\neq i} A^{i'}_\xi(m)$. Here this condition amounts to A^{as}-linearity of φ along the ith variable, and we are done. $\square$

3.4.11. Assume now that our A is commutative. Then the algebras A^{as}_x, A^{as} are also commutative. We already met A^{as}_x in situation when x is a single point, $x \in X(k)$ (see 2.4.8). The "geometric" description of A^{as}_x from 2.4.9 generalizes immediately to the general situation. Namely, for $x \in X(Y)$ as in 3.4.2 and a commutative quasicoherent $\mathcal{O}_Y$-algebra F we define a *non-quasicoherent* $\mathcal{O}_{X\times Y}$-algebra $O_x \hat{\otimes} F$ as the formal completion of $\mathcal{O}_X \boxtimes F$ at (the graph) of x; let $K_x \hat{\otimes} F$ be its localization with respect to an equation of x. Our $O_x \hat{\otimes} F \subset K_x \hat{\otimes} F$ are $\mathcal{D}_{X\times Y/Y}$-algebras, i.e., they carry an obvious connection along X. Now one has a canonical identification

$$\mathrm{Hom}(A^{as}_x, F) = \mathrm{Hom}(A^{\ell} \boxtimes \mathcal{O}_Y, K_x \hat{\otimes} F) = \mathrm{Hom}(j_{x*} j^*_x (A^{\ell} \boxtimes \mathcal{O}_Y), K_x \hat{\otimes} F) \tag{3.4.11.1}$$

where the first Hom means continuous morphisms of $\mathcal{O}_Y$-algebras and the next ones mean

[135] Here $\tilde{I} := I \sqcup \cdot$.

morphisms of $\mathcal{D}_{X\times Y/Y}$-algebras. For the construction see the proof of 2.4.9. As we mentioned in Remark of 2.4.9, (3.4.11.1) is the subset of algebra morphisms in (3.5.4.1).

3.4.12 Lemma. If $\varphi : A^\ell \to B^\ell$ is a formally smooth morphism of commutative $\mathcal{D}_X$-algebras then the corresponding morphism of topological $\mathcal{O}_Y$-algebras $A_x^{as} \to B_x^{as}$ is formally smooth. Same is true if we replace "formally smooth" by "formally étale".[136]

Proof. Assume that φ is formally smooth. Let F be a commutative k-algebra, $I \subset F$ an ideal such that $I^2 = 0$. Let $f : \mathrm{Spec}F \to \mathrm{Spec}A_x^{as}$, $g : \mathrm{Spec}F/I \to \mathrm{Spec}B_x^{as}$ be φ-compatible morphisms.[137] We want to find a lifting $h : \mathrm{Spec}F \to \mathrm{Spec}B_x^{as}$ of f that extends g.

According to (3.4.11.1) we may rewrite f, g as morphisms of $\mathcal{D}_{X\times Y/Y}$-algebras $f' : A^\ell \boxtimes \mathcal{O}_Y \to K_x \hat{\otimes} F$, $g' : B^\ell \boxtimes \mathcal{O}_Y \to K_x \hat{\otimes} F/I$. Here $K_x \hat{\otimes} F$, $K_x \hat{\otimes} F/I$ are non-quasicoherent sheaves of $\mathcal{D}_{X\times Y/Y}$-algebras.

Our problem is local (see [RG]) so we can assume that X and Y are affine. Denote by Γ the functor of global sections over $X \times Y$. So we have commutative $\mathcal{O}(X)$-algebras $\Gamma(K_x \hat{\otimes} F)$, $\Gamma(K_x \hat{\otimes} F/I)$ equipped with a connection along X. The second algebra is a quotient of the first one modulo an ideal of square 0.

Our f', g' amount to maps of commutative $\mathcal{O}(X)$-algebras $\Gamma f' : A^\ell(X) \to \Gamma(K_x \hat{\otimes} F)$, $\Gamma g' : B^\ell(X) \to \Gamma(K_x \hat{\otimes} F/I)$ compatible with connections along X. Since φ is a formally smooth morphism of $\mathcal{D}_X$-algebras we can find a lifting $\Gamma h' : B^\ell(X) \to \Gamma(K_x \hat{\otimes} F)$ of $\Gamma f'$ that extends $\Gamma g'$. As above, $\Gamma h'$ is the same as a morphism $h : \mathrm{Spec}F \to \mathrm{Spec}B_x^{as}$ which is the desired lifting.

The case of formally étale φ is left to the reader. □

§5 Chiral algebras as factorization algebras.

In this section we give another definition of chiral algebras in terms of $\mathcal{O}$-modules on the space of finite subsets of X which satisfy certain factorization property for disjoint

[136] For the definition of formally smooth and formally étale morphisms of commutative $\mathcal{D}_X$-algebras (or, more generally, algebraic $\mathcal{D}_X$-spaces) see 2.3.17.

[137] $\mathrm{Spec}A_x^{as}, \mathrm{Spec}B_x^{as}$ are ind-schemes.

union of subsets. This approach is technically very convenient (see §§7, 11). Also some important chiral algebras of geomeric nature do arise in this manner.

3.5.1 A topological digression. This subsection plays an important but purely motivational role.

Let X be a topological space. Its *exponential* $\mathcal{E}xp(X)$ is the set of all its finite subsets; for $S \subset X$ the corresponding point of $\mathcal{E}xp(X)$ is denoted by $[S]$. For any finite index set I we have an obvious map $r_I : X^I \to \mathcal{E}xp(X)$. We equip $\mathcal{E}xp(X)$ with the strongest topology such that all maps r_I are continuous. Thus $\mathcal{E}xp(X)$ is the disjoint union of the base point $[\emptyset]$ and the subspace $\mathcal{R}(X)$ called *Ran's space*[138] associated to X.

Here are some properties of $\mathcal{E}xp$ and $\mathcal{R}$:

(i) The subspaces $\mathcal{E}xp(X)_n$ that consists of $[S]$ with $|S| \leq n$ form an increasing filtration on $\mathcal{E}xp(X)$; the projection $r_n : X^n \to \mathcal{E}xp(X)_n$ is an open map. Same for $\mathcal{E}xp$ replaced by $\mathcal{R}$. One has $\mathcal{R}(X)_0 = \emptyset$, $\mathcal{R}(X)_1 = X$, $\mathcal{R}(X)_2 = \mathrm{Sym}^2 X$. A stratum $\mathcal{E}xp(X)^o_n := \mathcal{E}xp(X)_n \smallsetminus \mathcal{E}xp(X)_{n-1}$ (the space of configurations of n points in X) equals the complement to diagonals in $\mathrm{Sym}^n X$.

(ii) For any surjection $J \twoheadrightarrow I$ we have the corresponding diagonal embedding $\Delta^{(J/I)} : X^I \to X^J$. One has $r_J \Delta^{(J/I)} = r_I$, and $\mathcal{E}xp(X)$ is the inductive limit of the topological spaces X^I with respect to all diagonal embeddings. Same for $\mathcal{R}(X)$ (add the condition $I \neq \emptyset$).

(iii) $\mathcal{E}xp(X)$ is a commutative monoid, and $\mathcal{R}(X)$ is a commutative semigroup with respect to the operation $[S] \circ [S'] := [S \cup S']$. The map $\mathcal{E}xp(X_1) \times \mathcal{E}xp(X_2) \to \mathcal{E}xp(X_1 \sqcup X_2)$, $([S_1], [S_2]) \mapsto [S_1] \circ [S_2] = [S_1 \sqcup S_2]$, is a homeomorphism.

(iv) **Proposition.** All the homotopy groups of $\mathcal{E}xp(X)$ vanish. So if X is a connected CW complex then $\mathcal{R}(X)$ is contractible.

Remark. This result (in fact, a stronger one) is due to Curtis and Nhu (see Lemmas 3.3 and 3.6 from [CN]); simple-connectedness of $\mathcal{R}(X)_3$ was proved by Bott [Bo] who showed that $\mathcal{R}(S^1)_3 = S^3$. An algebraic proof, similar to given below, was communicated to us by

[138] $\mathcal{R}(X)$ was introduced by Borsuk and Ulam [BU]; Ziv Ran [R] was, probably, first to consider it in algebraic geometry.

Jacob Mostovoy.

Proof of the proposition. Below we write the group law on homotopy groups π_i, $i \geq 1$, multiplicatively. Let $v \in \mathcal{E}xp(X)$ be a base point. Consider the maps

$$\mathcal{E}xp(X) \xrightarrow{\delta} \mathcal{E}xp(X) \times \mathcal{E}xp(X) \xrightarrow{\circ} \mathcal{E}xp(X) \tag{3.5.1.1}$$

where δ is the diagonal embedding; one has $\circ\delta = id_{\mathcal{E}xp(X)}$. For $\gamma \in \pi_i(\mathcal{E}xp(X), v)$ one has $\delta(\gamma) = i_1(\gamma) \cdot i_2(\gamma) \in \pi_i(\mathcal{E}xp(X) \times \mathcal{E}xp(X), (v, v))$ where $i_1, i_2 : \mathcal{E}xp(X) \hookrightarrow \mathcal{E}xp(X) \times \mathcal{E}xp(X)$ are the embeddings $[S] \mapsto ([S], v)$, $(v, [S])$. Therefore $\gamma = \circ\delta(\gamma) = (v \circ \gamma)^2$. We see that $\gamma = 1$ if $v \circ \gamma = 1$. Notice that $v \circ v \circ \gamma = v \circ \gamma$, so, replacing in the previous formula γ by $v \circ \gamma$, we get $v \circ \gamma = (v \circ \gamma)^2$. Thus $v \circ \gamma = 1$, so $\gamma = 1$, and we are done. $\square$

Remark. For $v = [S] \in \mathcal{E}xp(X)$ let $\mathcal{E}xp(X, v) \subset \mathcal{E}xp(X)$ be the subspace whose points are finite subsets of X that contain S. The same argument as above shows that all homotopy groups of $\mathcal{E}xp(X, v)$ vanish. This also follows from the existence of a retraction $\rho : \mathcal{E}xp(X) \to \mathcal{E}xp(X, v)$ (put $\rho(u) = u \circ v$).

A *factorization algebra* is a sheaf B of vector spaces on $\mathcal{E}xp(X)$ (or, may be, an object of the appropriate derived category) together with natural identifications $B_{[S_1]} \otimes B_{[S_2]} \xrightarrow{\sim} B_{[S_1]\circ[S_2]}$ defined for *disjoint* S_1, S_2, which are associative and commutative in the obvious manner. Denote by B_X the restriction of B to $X \subset \mathcal{E}xp(X)$. The restriction of B to $\mathcal{E}xp(X)_i^o$ equals to the restriction of $\mathrm{Sym}^i(B_X)$ to $\mathrm{Sym}^i X \smallsetminus$(diagonals). A morphism between factorization algebras is completely determined by its restriction to B_X's, so one may consider B as a sheaf B_X on X equipped with certain *factorization* structure. This structure has X-local nature. So one may speak of factorization algebras *on* X.

3.5.2. We want to play with factorization algebras in the algebro-geometric setting of quasicoherent sheaves. From now on X is our curve. Notice that $\mathcal{R}(X)_i$ are *not* algebraic varieties for $i \geq 3$.[139] Nevertheless, we will consider $\mathcal{O}$- and $\mathcal{D}$-modules on $\mathcal{R}(X)$; they will be introduced directly using 3.5.1(ii).

[139] Exercise: show that every germ of a holomorphic function on $\mathcal{R}(X)_3$ at a point $x \in X \subset \mathcal{R}(X)_3$ is constant on X.

An $\mathcal{O}$-*module on* $\mathcal{R}(X)$, or an $\mathcal{O}_{\mathcal{R}(X)}$-*module*, is a rule F that assigns to every finite non-empty I a quasicoherent $\mathcal{O}_{X^I}$-module F_{X^I} and to every $\pi : J \twoheadrightarrow I$ an identification $\nu^{(\pi)} : \Delta^{(\pi)*} F_{X^J} \xrightarrow{\sim} F_{X^I}$ compatible with composition of π's. We demand that F_{X^I} have no non-zero local sections supported at the diagonal divisor.

$\mathcal{O}_{\mathcal{R}(X)}$-modules form a k-category $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$. This category is exact in the sense of [QK].[140] It is also a tensor k-category in the obvious way. The unit object is $\mathcal{O}_{\mathcal{R}(X)}$ (:= the $\mathcal{O}$-module with components $\mathcal{O}_{X^I}$).

Replacing in the above definition $\mathcal{O}$-modules by left $\mathcal{D}$-modules we get the notion of left $\mathcal{D}$-module on $\mathcal{R}(X)$. These objects form an exact tensor k-category $\mathcal{M}^{\ell}(\mathcal{R}(X))$ with unit object $\mathcal{O}_{\mathcal{R}(X)}$.

3.5.3 Lemma. Let F be an $\mathcal{O}_{\mathcal{R}(X)}$-module, J a finite non-empty set.

(i) F_{X^J} is flat along every diagonal $X^I \hookrightarrow X^J$, i.e., $\mathcal{T}or_i(F_{X^J}, \mathcal{O}_{X^I}) = 0$ for $i > 0$.

(ii) Let $\ell : V \hookrightarrow X^J$ be the complement to the diagonal strata of codimension ≥ 2. Then the map $F_{X^J} \to \ell_* \ell^* F_{X^J}$ is an isomorphism.[141]

Proof. (i) Our statement amounts to to the claim that for every $\pi : J \twoheadrightarrow I$ the projection $L\Delta^{(\pi)*} F_{X^J} \to \Delta^{(\pi)*} F_{X^J}$ is an isomorphism. Let $\tilde{\nu}^{(\pi)} : L\Delta^{(\pi)*} F_{X^J} \to F_{X^I}$ be its composition with the structure isomorphism $\nu^{(\pi)}$. Our $\tilde{\nu}^{(\pi)}$'s are compatible with composition of π's, so (since every diagonal embedding is composition of diagonal embeddings of codimension 1) it suffices to treat the codimension 1 case. Here our statement amounts to the property that F_{X^J} has no sections supported on X^I, and we are done.

(ii) We know that our map is injective. For every diagonal $X^I \subset X^J$ of codimension ≥ 2 one has $\mathcal{E}xt^1(\mathcal{O}_{X^I}, F_{X^J}) = 0$ by (i) (indeed, $\mathcal{E}xt^j(\mathcal{O}_{X^I}, F_{X^J})$ is locally isomorphic to $\mathcal{T}or_{|J|-|I|-j}(\mathcal{O}_{X^I}, F_{X^J})$[142]). This implies surjectivity of our map. Namely, take the smallest m such that $C := \mathrm{Coker}(F_{X^J} \to \ell_* \ell^* F_{X^J})$ vanishes outside of diagonals of dimension $\leq m$. Then for a diagonal $X^I \subset X^J$ of dimension m our C is non-zero on $X^I \smallsetminus$ (diagonals), hence $\mathcal{H}om(\mathcal{O}_{X^I}, C) \neq 0$. The latter sheaf equals $\mathcal{E}xt^1(\mathcal{O}_{X^I}, F_{X^J})$, and we are done. $\square$

[140] A short sequence is exact if and only if all the corresponding sequences of $\mathcal{O}_{X^n}$-modules are exact. We do not know if $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$ is an abelian category.

[141] Here ℓ_* is the naive sheaf-theoretic direct image (no higher derived functors are considered).

[142] Compute them using a Koszul resolution of $\mathcal{O}_{X^I}$.

3.5.4. For $\pi : J \twoheadrightarrow I$ let $j^{[J/I]} : U^{[J/I]} \hookrightarrow X^J$ be the complement to all the diagonals that are transversal to $\Delta^{(J/I)} : X^I \hookrightarrow X^J$. So $U^{[J/I]} = \{(x_j) : x_{j_1} \neq x_{j_2} \text{ if } \pi(j_1) \neq \pi(j_2)\}$.

Let B be an $\mathcal{O}$-module on $\mathcal{R}(X)$. A *factorization structure* on B is a rule that assigns to every $J \twoheadrightarrow I$ an isomorphism of $\mathcal{O}_{U^{[J/I]}}$-modules

$$\kappa_{[J/I]} : j^{[J/I]*}(\underset{I}{\boxtimes} B_{X^{J_i}}) \xrightarrow{\sim} j^{[J/I]*} B_{X^J}. \tag{3.5.4.1}$$

We demand that κ's are mutually compatible: for $K \twoheadrightarrow J$ the isomorphism $\kappa_{[K/J]}$ coincides with the composition $\kappa_{[K/I]}(\boxtimes \kappa_{[K_i/J_i]})$.

Notice that $\kappa_{[J/J]}$ identifies $j^{(J)*} B_{X^J}$ with $j^{(J)*} B_X^{\boxtimes J}$, so we have a canonical embedding

$$B_{X^J} \hookrightarrow j^{(J)*} B_X^{\boxtimes J}. \tag{3.5.4.2}$$

$\mathcal{O}$-modules on $\mathcal{R}(X)$ equipped with factorization structure form a tensor category in the obvious manner.[143]

As follows from 3.5.4.2 the functor $B \mapsto B_X$ is faithful. Therefore we can consider an $\mathcal{O}$-module B on $\mathcal{R}(X)$ equipped with a factorization structure as an $\mathcal{O}$-module B_X on X equipped with certain structure which we also call factorization structure. Notice that the factorization structure has X-local nature, so $\mathcal{O}$-modules with factorization structure form a sheaf of tensor categories on the étale topology of X.

In the above discussion one may replace everywhere $\mathcal{O}$-modules by left $\mathcal{D}$-modules.

Remark. If an $\mathcal{O}_{\mathcal{R}(X)}$-module B admits a factorization structure then B is flat if (and only if) B_X is a flat $\mathcal{O}_X$-module.[144]

3.5.5 Definition. A *factorization algebra* B on X is an $\mathcal{O}$-module equiped with a factorization structure which satisfies the following condition:

There exists a global section 1 of B_X such that for every $f \in B_X$ one has $1 \boxtimes f \in B_{X^2} \subset j_* j^* B_X^{\boxtimes 2}$ and $\Delta^*(1 \boxtimes f) = f$.

[143] This category is not additive.

[144] Proof: Assume that B_X is flat. We want to show that every B_{X^n} is flat. It amounts to vanishing of $Tor_{>0}(B_{X^n}, K)$ for all morphisms $\phi : \mathrm{Spec} K \to X^n$, K is a field. Using induction by n and 3.5.3(i) we can assume that $\phi(\mathrm{Spec} K) \subset U^{(n)}$, and $B_{U^{(n)}}$ is flat by factorization.

Notice that such section 1 is uniquely defined; it is called *the unit* in B.

Factorization algebras form a tensor category[145] which we denote by $\mathcal{FA}(X)$. They are objects of X-local nature, so we have a sheaf of categories $\mathcal{FA}(X_{\text{ét}})$ on the étale topology of X.

The unit object of our tensor category is the "trivial" factorization algebra $\mathcal{O}$. For any $B \in \mathcal{FA}(X)$ there is a unique morphism $\mathcal{O} \to B$. Therefore for a collection B_α of factorization algebras and $B := \otimes B_\alpha$ one has canonical morphisms $\nu_\alpha : B_\alpha \to B$.[146]

Remark. In the category $\mathcal{FA}(X)$ infinite tensor products are well-defined.[147] Namely, if B_α is any collection of factorization algebras then $\otimes B_\alpha$ is the inductive limit of $\underset{\alpha\in S}{\otimes} B_\alpha$ where S runs the set of finite subsets of indices α. Here for $S \subset S'$ the corresponding arrow $\underset{\alpha\in S}{\otimes} B_\alpha \to \underset{\alpha\in S'}{\otimes} B_\alpha$ is $\otimes b_\alpha \mapsto (\otimes b_\alpha) \otimes (\underset{\alpha'\in S'\smallsetminus S}{\otimes} 1'_\alpha)$.

3.5.6. Here is another description of factorization algebras. The reader may skip it and pass directly to 3.5.8.

For a scheme Z let $\mathcal{C}(X)_Z$ be the set of relative effective Cartier divisors in $X \times Z$ over Z. This is a commutative monoid (a submonoid in the group of all Cartier divisors). The usual ordering of Cartier divisors makes $\mathcal{C}(X)_Z$ a category. This is a tensor category with respect to the usual addition of divisors. Our $\mathcal{C}(X)_Z$ form a category $\mathcal{C}(X)$ fibered over the category of schemes Z; this is a sheaf of categories with respect to the flat topology.

Equivalently, objects of $\mathcal{C}(X)_Z$ are subschemes $S \subset X \times Z$ finite, flat, and of finite presentation over Z (see [KM], 1.2.3); the morphisms are embeddings of S's.

Remark. For any $n \geq 0$ the subcategory $\mathcal{C}(X,n)_Z \subset \mathcal{C}(X)_Z$ of divisors of degree n is discrete. The obvious divisor $\in C(X)_{\mathrm{Sym}^n X}$ is universal (the functor $Z \mapsto \mathcal{C}(X,n)_Z$ is representable by $\mathrm{Sym}^n X$).

Consider a pair (B, κ) where:

(i) B is a Morphism from the fibered category $\mathcal{C}(X)$ to that of $\mathcal{O}$-modules. Thus B is a rule that assigns to every $S \in \mathcal{C}(X)$ a quasicoherent $\mathcal{O}_Z$-module $B(S) = B(S)_Z$; for $S \subset T$

[145] We demand that the morphisms preserve units.

[146] Defined as tensor product of id_{B_α} and the canonical morphisms $\mathcal{O} \to B_{\alpha'}$, $\alpha' \neq \alpha$.

[147] Just as in the usual category of *unital* associative algebras.

we have a canonical morphism $B(S) \to B(T)$; everything is compatible with base change. In particular, the universal divisors yield $\mathcal{O}$-modules $B_{\mathrm{Sym}^n X}$ on $\mathrm{Sym}^n X$.

(ii) κ is a rule that assigns to every pair of *mutually disjoint* divisors $S_1, S_2 \in \mathcal{C}(X)_Z$ an identification $\kappa_{S_1,S_2} : B(S_1) \otimes B(S_2) \xrightarrow{\sim} B(S_1 + S_2)$. We demand that these identifications are commutative and associative in the obvious manner, and are compatible with the natural morphisms from (i). So for any finite family of mutually disjoint divisors S_α there is a canonical identification

$$\kappa_{\{S_\alpha\}} : \otimes B(S_\alpha) \xrightarrow{\sim} B(\Sigma S_\alpha). \tag{3.5.6.1}$$

Assume that the following conditions hold:

(a) For every $S, S' \in \mathcal{C}(X)_Z$ such that $S \subset S'$ and $S_{red} = S'_{red}$ the canonical morphism $B(S) \to B(S')$ is an isomorphism.

(b) The $\mathcal{O}$-modules $B_{\mathrm{Sym}^n X}$ on $\mathrm{Sym}^n X$ have no non-zero local sections supported at the diagonal divisor.

(c) One has $B(\emptyset) \neq 0$.[148]

We are going to show that such (B, κ) is the same as a factorization algebra. For the moment we call such object a factorization algebra in the sense of 3.5.6. They form a category which we denote by $\mathcal{FA}(X)'$. This is a tensor category in the obvious manner (one has $(\otimes B_i)(S) = \otimes(B_i(S))$).

Let (B, κ) be a factorization algebra in the sense of 3.5.6. Let us define the corresponding factorization algebra in the sense of 3.5.5 which we denote again by B. For any $I \in \mathcal{S}$ the points $x = x_i$, $i \in I$, form a Cartier divisor of degree $|I|$ in $X \times X^I/X^I$. So we have the corresponding $\mathcal{O}$-module B_{X^I} on X^I (it equals to the pull-back of $B_{\mathrm{Sym}^{|I|} X}$ by the projection $p_{|I|} : X^I \to \mathrm{Sym}^{|I|} X$). These $\mathcal{O}$-modules form an $\mathcal{O}$-module on $\mathcal{R}(X)$ in the sense of 3.5.2: indeed, the pull-back compatibility of B together with (a) provide the isomorphisms $\nu^{(\pi)}$, and (b) assures that B_{X^I} have no sections supported on the diagonal divisors.

[148] As follows from 3.5.6.1 this property just excludes B identically equal to 0.

The $\mathcal{O}$-module on $\mathcal{R}(X)$ we have defined carries a factorization structure (see 3.5.4) defined by isomorphisms 3.5.6.1. It has a unit section 1 (see 3.5.5). To define 1 look at the canonical isomorphism $\kappa_\emptyset : B(\emptyset)^{\otimes 2} \xrightarrow{\sim} B(\emptyset)$. Then $B(\emptyset)$ is a one dimensional vector space[149] generated by an element 1 such that $\kappa_\emptyset(1 \otimes 1) = 1$. Since $\emptyset$ lies in any divisor our 1 yields a section of every $B(S)$, in particular, of B_{X^I}. The property from 3.5.5 is immediate.

Our construction is obviously functorial and compatible with tensor products, so we have defined a canonical tensor functor

$$\mathcal{FA}(X)' \to \mathcal{FA}(X). \tag{3.5.6.2}$$

3.5.7 Proposition. This functor is an equivalence of categories.

Proof. We will construct the inverse functor. Let B be a factorization algebra in the sense of 3.5.5. We want to define the corresponding factorization algebra (B, κ) in the sense of 3.5.6.

First let us construct the $\mathcal{O}$-modules $B_{\mathrm{Sym}^n X}$. One has $\mathrm{Sym}^0 X = \mathrm{Spec} k$ and $B(\emptyset) = k$. For $n > 0$ we consider the projection $p_n : X^n \to \mathrm{Sym}^n X$. The symmetric group Σ_n acts on X^n and on B_{X^n}. Set $B_{\mathrm{Sym}^n X} := (p_{n*} B_{X^n})^{\Sigma_n}$. By 3.5.2 it satisfies condition 3.5.6(b).

Let us define $B(S)_Z$ for every $S \in \mathcal{C}(X)_Z$. Our problem is Z-local, so we may assume that the degree n of S over Z is constant. Such S amounts to a morphism $Z \to \mathrm{Sym}^n X$, and $B(S)_Z$ is the pull-back of $B_{\mathrm{Sym}^n X}$.

Lemma. The canonical morphism $\psi : p_n^* B_{\mathrm{Sym}^n X} \to B_{X^n}$ is an isomorphism.

Proof of the lemma. (a) Let $\ell : V \hookrightarrow X^n$ be the complement to the diagonals of codimension ≥ 2 and $\bar{\ell} : \bar{V} \hookrightarrow \mathrm{Sym}^n X$ be its p_n-image. By 3.5.3(ii) we know that $B_{X^n} = \ell_* \ell^* B_{X^n}$, thus $B_{\mathrm{Sym}^n X} = \bar{\ell}_* \bar{\ell}^* B_{\mathrm{Sym}^n X}$. Since p_n is flat we conclude that it suffices to check that ψ is an isomorphism over V. By factorization it suffices to consider the case $n = 2$.

(b) By étale descent our ψ is an isomorphism over the complement to the diagonal divisor. Therefore ψ is injective (since p_n is flat).

[149] Our $\kappa_\emptyset$ is commutative. It is also injective, so for any $a, b \in B(\emptyset)$ one has $a \otimes b = b \otimes a \in B(\emptyset)^{\otimes 2}$. Together with (c) this implies that $\dim B(\emptyset) = 1$. Exercise: deduce the same fact from associativity of $\kappa_\emptyset$.

(c) It remains to prove that $\psi : p_2^* B_{\mathrm{Sym}^2 X} \to B_{X^2}$ is surjective. Consider its cokernel C. This is a Σ_2-equivariant quotient of B_{X^2} such that $(p_{2*}C)^{\Sigma_2} = 0$. We know that C is supported (set-theoretically) on the diagonal divisor X, so one has $C^{\Sigma_2} = 0$.

Let f be a (local) equation of the diagonal $X \subset X \times X$ which is Σ_2-antiinvariant. Since Σ_2 acts on C by the non-trivial character sgn multiplication by f kills C, i.e., C is supported on the diagonal scheme-theoretically. Thus C is a quotient of the pull-back of B_{X^2} to X, i.e., of B_X. But Σ_2 acts on B_X trivially. So $C = 0$, and we are done. □

Remark. Let $G \subset \Sigma_n$ be a subgroup, so we have the projections $X^n \xrightarrow{\pi_1} G \setminus X^n \xrightarrow{\pi_2} \mathrm{Sym}^n X$. The lemma implies that $\pi_2^* B_{Sym^n X} = (\pi_{1*} B_{X^n})^G$.

To get the structure from 3.5.6(i) it remains to define for every $S, T \in \mathcal{C}(X)_Z$ such that $S \subset T$ a canonical morphism $B(S) \to B(T)$. We have $T = S + S'$ for an effective Cartier divisor S'. It suffices to define our morphism in the universal situation when $Z = \mathrm{Sym}^n X \times \mathrm{Sym}^m X$ and S, S' are the universal divisors of order n and m respectively. So let $q^{(n)} : \mathrm{Sym}^n X \times \mathrm{Sym}^m X \to \mathrm{Sym}^n X$ be the projection, $p : \mathrm{Sym}^n X \times \mathrm{Sym}^m X \to \mathrm{Sym}^{n+m} X$ the addition map. We want to define a canonical morphism $q^{(n)*} B_{Sym^n X} \to p^* B_{\mathrm{Sym}^{n+m} X}$. By the above lemma and remark this amounts to a $\Sigma_n \times \Sigma_m$ -equivariant morphism $B_{X^n} \boxtimes \mathcal{O}_{X^m} \to B_{X^{n+m}}$. By definition, our morphism sends $b \boxtimes f \in B_{X^n} \boxtimes \mathcal{O}_{X^m}$ to the section $\kappa(b \boxtimes f 1_B^{\boxtimes m})$. Here κ is the factorization isomorphism (3.5.4.1), and our section belongs to $B_{X^{n+m}}$ (see (3.5.4.2)) by 3.5.3(ii) and the definition of the unit 1_B.

The canonical morphism $B(S) \to B(T)$ so defined is compatible with composition of embeddings $S \subset T$. So we have defined the structure 3.5.6(i). As for the factorization isomorphism 3.5.6(ii), it is defined by the factorization structure (3.5.4.1) in the obvious manner (consider the universal situation and use the above remark).

We leave it to the reader to check the condition (a) of 3.5.6. So we have defined the factorization algebra in the sense of 3.5.6, i.e., we have a functor $\mathcal{FA}(X) \to \mathcal{FA}(X)'$. It follows from the construction (use the lemma) that it is inverse to (3.5.6.2). □

3.5.8 Proposition. Let B be a factorization algebra. Then the $\mathcal{O}$-module B on $\mathcal{R}(X)$ admits a left $\mathcal{D}$-module structure compatible with the factorization structure and such

that the section 1 of B_X is horizontal. Such $\mathcal{D}$-module structure is unique.

We call it *the canonical* $\mathcal{D}$-module structure on B.

Proof. Let Y be any smooth variety. For a finite non-empty set I denote by $Y^{<I>}$ the formal completion of Y^I at the diagonal $Y \subset Y^I$. A map $\pi : J \to I$ yields the morphism $\Delta^{(\pi)} : Y^I \to Y^J$, $(y_i) \mapsto (y_j)$, $y_j = y_{\pi(j)}$, hence the morphism of formal schemes $\Delta^{<\pi>} : Y^{<I>} \to Y^{<J>}$. According to Grothendieck, a left $\mathcal{D}$-module on Y amounts to a rule which assigns to every I an $\mathcal{O}$-module $F_{Y^{<I>}}$ on $Y^{<I>}$ and to every $\pi : J \to I$ an isomorphism $\nu^{<\pi>} : \Delta^{<\pi>*} F_{Y^{<J>}} \xrightarrow{\sim} F_{Y^{<I>}}$ compatible with composition of π's.[150] Namely, there is a unique way to define on each $F_{Y^{<I>}}$ a left action of $\mathcal{D}_{Y^I}$ so that all $\nu^{<\pi>}$ become isomorphisms of $\mathcal{D}$-modules. The corresponding $\mathcal{D}$-module on Y is F_Y equipped with this $\mathcal{D}$-module structure.

We define our left $\mathcal{D}$-module structure on B_X using this language. Key point: every map $\pi : J \to I$ yields a canonical morphism

$$\nu^{(\pi)} : \Delta^{(\pi)*} B_{X^J} \to B_{X^I}. \tag{3.5.8.1}$$

Namely, if we use the interpretation of factorization algebras from 3.5.6 then $\nu^{(\pi)}$ is just the structure morphism from 3.5.6(i). If the reader prefers to avoid 3.5.6, 3.5.7 then he may define the $\nu^{(\pi)}$'s directly: this is the unique family of morphisms compatible with composition of π's such that for π surjective $\nu^{(\pi)}$ is the structure isomorphism from 3.5.2, and for π injective our $\nu^{(\pi)} : \Delta^{(\pi)*} B_{X^J} = B_{X^J} \boxtimes \mathcal{O}_{X^{I \smallsetminus J}} \to B_{X^I}$ is $b \boxtimes f \mapsto \kappa(b \boxtimes f 1^{\boxtimes I \smallsetminus J})$. Here κ is the factorization isomorphism (3.5.4.1); to see that the latter section belongs to B_{X^I} (see (3.5.4.2)) use 3.5.3(ii) and the definition of the unit in B (see 3.5.5).

Now we are ready to define the canonical left $\mathcal{D}$-module structure on B. Let $B_{X^{<I>}}$ be the pull-back of B_{X^I} to $X^{<I>}$ (i.e., the completion of B_{X^I} at the diagonal $X \subset X^I$), $\nu^{<\pi>}$ be the pull-back of the morphism 3.5.8.1 to $X^{<I>}$. Our $\nu^{<\pi>}$ is actually an isomorphism: this follows from 3.5.3(i) and the fact that the pull-back of $\nu^{(\pi)}$ to the diagonal $X \subset X^I$ is an isomorphism (equal to id_{B_X}).

[150] We assume that F_Y is a quasicoherent $\mathcal{O}_Y$-module.

We have defined the $\mathcal{D}$-module structure on B_X. One defines a $\mathcal{D}$-module structure on each B_{X^S} in the similar way taking for $B_{(X^S)<I>}$ the pull-back of $B_{X^{S\times I}}$ to the formal neighbourhood of $X^S \subset (X^S)^I = X^{S\times I}$, and for $\nu^{<\pi>}$ the pull-back of $\nu^{(id_S\times\pi)}$.

One checks immediately that the morphisms $\nu^{(\pi)}$ are compatible with the $\mathcal{D}$-module structure, as well as the morphisms κ from 3.5.4.1, so we have defined a left $\mathcal{D}$-module structure on B. The section 1 is obviously horizontal, as required.

Let us prove the uniqueness statement. Assume we have another $\mathcal{D}$-module structure on B such that 1 is horizontal. This property implies that every $\nu^{(\pi)}$ is compatible with our $\mathcal{D}$-module structure. Pulling it back to the formal neighbourhood of the diagonals we get a $\mathcal{D}$-module structure on every $B_{X<I>}$ (and $B_{(X^S)<I>}$) such that the $\nu^{<\pi>}$'s are morphisms of $\mathcal{D}$-modules. According to the uniqueness statement in the first paragraph of our proof, this datum of $\mathcal{D}$-module structures is uniquely defined. Hence the $\mathcal{D}$-module structures on B_{X^S} are also uniquely defined (set $I = \cdot$), and we are done. □

Remarks. (i) Here is a down-to-earth definition of the canonical $\mathcal{D}$-module structure (which follows directly from the proof of 3.5.8). Let t be a local coordinate on X. Then for $b \in B$ one has

$$1 \boxtimes b - b \boxtimes 1 = (\partial_t b) \boxtimes 1(t_2 - t_1)\mathrm{mod}(t_2 - t_1)^2 B_{X\times X}. \tag{3.5.8.2}$$

In fact, the restriction of $1 \boxtimes b$ to the formal neighbourhood of the diagonal $X \subset X \times X$ equals $\sum \frac{(t_2-t_1)^i}{i!}(\partial_t^i b) \boxtimes 1$. This defines our $\mathcal{D}$-module structure on B_X. And B_{X^I} is a $\mathcal{D}$-submodule of $j_*^{(I)} j^{(I)*} B_X^{\boxtimes I}$.

(ii) In the setting of 3.5.6 $\mathcal{O}$-modules $B_Z(S)$ (for smooth Z) have a canonical left $\mathcal{D}$-module structure uniquely determined by the property that all the structure morphisms from 3.5.6(i),(ii) are compatible with the $\mathcal{D}$-module structure. The proof and construction essentially coincides with the one from the above proposition, so we skip it. In particular, we see that the canonical $\mathcal{D}$-module structure on B_{X^n} comes from the one on $B_{\mathrm{Sym}^n X}$.

3.5.9. Let $B^r = B_X\omega_X$ be the right $\mathcal{D}_X$-module that corresponds to the left $\mathcal{D}_X$-module B_X. Let us show that it carries a canonical structure of chiral algebra.

For every $J \twoheadrightarrow I$ we have a canonical isomorphism of left $\mathcal{D}$-modules $\Delta^{(J/I)*}B_{X^J} \xrightarrow{\sim} B_{X^I}$. According to (2.1.3.1) we may rewrite it as an isomorphism of right $\mathcal{D}_{X^J}$-modules

$$B_{X^J} \otimes \Delta_*^{(J/I)}\omega_X^{\boxtimes I} \xrightarrow{\sim} \Delta_*^{(J/I)}(B_{X^I} \otimes \omega_X^{\boxtimes I}). \tag{3.5.9.1}$$

In particular, we get a canonical map

$$P^{ch}(\omega_X)_J \longrightarrow P^{ch}(B_X^r)_J \tag{3.5.9.2}$$

which sends an operation $\varphi : j_*^{(J)}j^{(J)*}\omega_X^{\boxtimes J} \longrightarrow \Delta_*^{(J)}\omega_X$ to the composition

$$j_*^{(J)}j^{(J)*}B^{r\boxtimes J} = B_{X^J} \otimes j_*^{(J)}j^{(J)*}\omega_X^{\boxtimes J} \longrightarrow B_{X^J} \otimes \Delta_*^{(J)}\omega_X = \Delta_*^{(J)}B^r. \tag{3.5.9.3}$$

Here the first equality is the factorization isomorphism tensored by $id_{\omega_X^{\boxtimes J}}$, the arrow is φ tensored by $id_{B_{X^J}}$, and the last equality is 3.5.9.1 (for $I = \cdot$).

We leave it to the reader to check that 3.5.9.2 is compatible with composition of chiral operations. According to 3.1.4 we may rewrite 3.5.9.2 as a canonical morphism of operads $\mathcal{L}ie \to P^{ch}(B^r)$ which is the same as a Lie^{ch} algebra structure on B_X^r. It is easy to see that the section 1 of B_X is a unit in the Lie^{ch} algebra B^r. So B^r is a chiral algebra.

The folowing theorem is the principal result of this section:

3.5.10 Theorem. The functor $\mathcal{FA}(X) \to \mathcal{CA}(X), \quad B \mapsto B^r$, is an equivalence of categories.

Proof. We will construct the inverse functor. So for every chiral algebra A we want to define on the left $\mathcal{D}_X$-module A^ℓ a canonical structure of factorization algebra. The construction makes use of certain canonical complexes associated to A. These complexes will also be used in 3.10.

3.5.11. For a finite non-empty I consider the following complex $C(A)_{X^I}^{\cdot}$ of right $\mathcal{D}$-modules on X^I. Let us forget about the differential for a moment, so we consider a plain $\mathbb{Z}$-graded $\mathcal{D}$-module $C_I(A)^{\cdot}$. One has

$$C(A)_{X^I}^{\cdot} := \bigoplus_{T \in Q(I)} \Delta_*^{(I/T)}j_*^{(T)}j^{(T)*}(A[1])^{\boxtimes T}. \tag{3.5.11.1}$$

In other words, $C(A)^{\cdot}_{X^I}$ is a $Q(I)$-graded module, and for $T \in Q(I)$ its T-component is $\Delta_*^{(I/T)} j_*^{(T)} j^{(T)*} A^{\boxtimes T} \otimes \lambda_T$ sitting in degree $-|T|$. Let us define now the differential. Its (T, T')-component $d_{T,T'} : \Delta_*^{(I/T)} j_*^{(T)} j^{(T)*}(A[1])^{\boxtimes T} \to \Delta_*^{(I/T')} j_*^{(T')} j^{(T')*}(A[1])^{\boxtimes T'}$ may be non-zero only for $T' \in Q(T, |T|-1)$. Then $T = T'' \sqcup \{\alpha', \alpha''\}$, $T' = T'' \sqcup \{\alpha\}$ and $d_{T,T'}$ is the exterior tensor product of the chiral product map $\mu_A[1] : j_* j^*(A_{\alpha'}[1] \boxtimes A_{\alpha''}[1]) \to \Delta_* A_\alpha[1]$ and the identity map for $A^{\boxtimes T''}$ (localized at the diagonals transversal to $X^{T'}$). The Jacobi identity for μ_A implies that $d^2 = 0$.

Notice that if $v : V \hookrightarrow X^I$ is an open embedding such that $X^I \smallsetminus V$ is a union of diagonal strata then the canonical map $C(A)_{X^I} \to v_* v^* C(A)_{X^I}$ admits an obvious canonical section

$$s : v_* v^* C(A)^{\cdot}_{X^I} \to C(A)^{\cdot}_{X^I} \tag{3.5.11.2}$$

which preserves the $Q(I)$-grading but does not commute with the differentials.

The complexes $C(A)_{X^I}$ enjoy the following two functorial properties:

(i) For every $\pi : J \twoheadrightarrow I$ there is an obvious identification of complexes

$$\Delta^{(\pi)!} C(A)_{X^J} = C(A)_{X^I}, \tag{3.5.11.3}$$

and this plain $\Delta^{(\pi)!}$ pull-back equals to the derived functor one; these identifications are compatible with composition of π's. In other words, one has a canonical short exact sequence of complexes

$$0 \to \Delta_*^{(\pi)} C(A)_{X^I} \to C(A)_{X^J} \to v_* v^* C(A)_{X^J} \to 0. \tag{3.5.11.4}$$

Here $v : X^J \smallsetminus \Delta^{(\pi)}(X^I) \hookrightarrow X^J$ is the open embedding complementary to $\Delta^{(\pi)}$; the naive v_* coincides here with the derived one.

(ii) (factorization) In the setting of 3.5.4 for $J \twoheadrightarrow I$ there is an obvious canonical isomorphism of complexes

$$\kappa_{[J/I]} : j^{[J/I]*}(\underset{I}{\boxtimes} C(A)_{X^{J_i}}) \xrightarrow{\sim} j^{[J/I]*} C(A)_{X^J}. \tag{3.5.11.5}$$

The identifications κ are mutually compatible: for $K \twoheadrightarrow J$ the isomorphism $\kappa_{[K/J]}$ coincides with the composition $\kappa_{[K/I]}(\boxtimes \kappa_{[K_i/J_i]})$.

3.5.12 Lemma. The complex $C(A)_{X^I}$ is acyclic outside degree $-|I|$.

Proof. The statement is clear if $|I| = 1$ (one has $C(A)_X = A[1]$). For $|I| > 1$ we use induction on $|I|$. The proof goes in 3 steps:

(i) The module $H^a C(A)_{X^I}$ for $a \neq -|I|$ is supported on the diagonal $X \subset X^I$. This follows from 3.5.11.5 and the induction assumption.

Let $i : Y = X^{I'} \hookrightarrow X^I$ be a diagonal of codimension 1, $v : X^I \smallsetminus Y \hookrightarrow X^I$ the complementary embedding. Consider the canonical embedding $\xi : i_* C(A)_Y \hookrightarrow C(A)_{X^I}$ (see 3.5.11.4).

(ii) The morphism $H^a\xi : i_* H^a C(A)_Y = H^a i_* C(A)_Y \to H^a C(A)_{X^I}$ is surjective for $a \neq -|I|$. Indeed, the complex $v_* v^* C(A)_{X^I}$ is acyclic outside degree $-|I|$ by (i) (since v is affine), so our claim follows from the exact sequence $0 \to i_* C(A)_Y \to C(A)_{X^I} \to v_* v^* C(A)_{X^I} \to 0$ (see 3.5.11.4).

(iii) It remains to show that $H^a\xi = 0$. If $a \neq 1 - |I|$ this is clear because in this case $H^a C(A)_Y = 0$ by the induction assumption.

Now let $a = 1 - |I|$. Choose a section $I' \to I$ of $I \twoheadrightarrow I'$, then we have the decomposition $X^I = X \times X^{I'} = X \times Y$ and the embedding $i : Y \hookrightarrow X \times Y$. We want to show that $\xi(i_* Z^a_Y) \subset d(C^{a-1}(A)_{X^I})$ where $Z^a_Y := \mathrm{Ker}(d : C^a(A)_Y \to C^{a+1}(A)_Y)$. Let s be a local section of $i_* Z^a_Y = v_* v^*(\omega_X \boxtimes Z^a_Y)/\omega_X \boxtimes Z^a_Y$. Lift it to a local section $\tilde{s}$ of $v_* v^*(\omega_X \boxtimes Z^a_Y)$. Then $\xi(s) = d(h(\tilde{s}))$ where h is the composition

$$v_* v^*(\omega_X \boxtimes Z^a_Y) \xrightarrow{f} v_* v^*(A \boxtimes C^a(A)_Y) \hookrightarrow C^{a-1}(A)_{X^I}$$

and f comes from $1_A : \omega_X \to A$. □

3.5.13. *End of the proof of* 3.5.10. Now we are ready to define the factorization algebra structure on A^ℓ. Set $A^\ell_{X^I} := H^{-|I|} C(A)^\ell_{X^I} = \mathrm{Ker}(d : C(A)^{-|I|}_{X^I} \to C(A)^{-|I|+1}_{X^I})^\ell$. By 3.5.12 we may rewrite 3.5.11.3 as a canonical identification $\Delta^{(I/S)*} A^\ell_{X^I} = A^\ell_{X^S}$ (see 2.1.2). These identifications are obviously transitive, so the $A^\ell_{X^I}$'s form a left $\mathcal{D}$-module, hence an $\mathcal{O}$-module, on $\mathcal{R}(X)$. So[151] $A^\ell_{X^I} \subset C(A)^{0\ell}_{X^I} = j^{(I)}_* j^{(I)*} A^{\ell \boxtimes I}$. We leave it to the reader to

[151] We use the identification $\omega_{X^I} = \omega_X^{\boxtimes I} \otimes \lambda_I$, see 3.1.3.

check that this embedding defines a factorization algebra structure on our $\mathcal{O}_{\mathcal{R}(X)}$-module (with the unit equal to the unit in A). Notice that our factorization algebra came together with a $\mathcal{D}$-module structure, which obviously coincides with the canonical one (see 3.5.8). One sees immediately that the functor $\mathcal{FA}(X) \to \mathcal{CA}(X)$ we have defined is inverse to that of 3.5.10. We are done. □

3.5.14 Remark. For $n \geq 1$ an *n-truncated factorization algebra* is defined exactly as usual factorization algebra but we take in consideration only the X^I's with $|I| \leq n$. Denote the corresponding categories by $\mathcal{FA}^{\leq n}(X)$. As follows from 3.5.10, for $n \geq 3$ the obvious functors $\mathcal{FA}(X) \to \mathcal{FA}^{\leq n}(X)$, $\mathcal{FA}^{\leq n+1}(X) \to \mathcal{FA}^{\leq n}(X)$ are equivalences of categories.[152]

3.5.15. We know that $\mathcal{FA}(X)$ is a tensor category (see 3.5.4, 3.5.5), hence such is $\mathcal{CA}(X)$. Let us explain the meaning of tensor product of chiral algebras in purely chiral terms.

Let A_α be a family of chiral algebras. Let $A := \otimes A_\alpha \in \mathcal{CA}(X)$ be the corresponding tensor product, so $A^\ell = \otimes A_\alpha^\ell$. By 3.5.5 we have canonical morphisms of chiral algebras $\nu_\alpha : A_\alpha \to A$, $a \mapsto a \otimes (\underset{\alpha' \neq \alpha}{\otimes} 1_{A_{\alpha'}})$.

Let $\varphi_\alpha : A_\alpha \to C$ be morphisms of chiral algebras. We say that φ_α *mutually commute* if $[\varphi_{\alpha'}, \varphi_{\alpha''}]_C \in P_2^*(\{A_{\alpha'}, A_{\alpha''}\}, C)$ vanishes for every $\alpha' \neq \alpha''$. Denote by $\mathrm{Hom}_I^{com}(\{A_\alpha\}, C)$ the set of all mutually commuting (φ_α)'s.

Remark. Let $M_\alpha \subset A_\alpha$ be a sub-$\mathcal{D}$-module that generates A_α as a chiral algebra. Then $\varphi_{\alpha'}$, $\varphi_{\alpha''}$ mutually commute if (and only if) their restrictions to M's commute, i.e., the image of $[\varphi_{\alpha'}, \varphi_{\alpha''}]_C$ in $P_2^*(\{M_{\alpha'}, M_{\alpha''}\}, C)$ vanishes.

3.5.16 Proposition. $\{\nu_\alpha\}$ is a universal family of mutually commuting morphisms: for every $C \in \mathcal{CA}(X)$ the map $\mathrm{Hom}(\otimes A_\alpha, C) \to \mathrm{Hom}_I^{com}(\{A_\alpha\}, C)$, $\psi \mapsto (\psi\nu_\alpha)$, is bijective.

Proof. Let us translate 3.5.16 back in the setting of factorization algebras. Assume for simplicity of notation that we play with two algebras A_1, A_2. Notice that φ_1, φ_2 as above mutually commute if and only if for every $a_1 \in A_1^\ell, a_2 \in A_2^\ell$ the section $\varphi_1(a_1) \boxtimes \varphi_2(a_2)$ of $j_* j^* C^\ell_{X \times X}$ lies in $C^\ell_{X \times X}$. One has $\nu_1(a_1) \boxtimes \nu_2(a_2) = (a_1 \boxtimes 1_{A_1}) \otimes (1_{A_2} \boxtimes a_2) \in A^\ell_{1X \times X} \otimes A^\ell_{2X \times X} = (A_1^\ell \otimes A_2^\ell)_{X \times X}$, so our $\nu_{1,2}$ mutually commute. For $\varphi_{1,2} \in \mathrm{Hom}_I^{com}(\{A_1, A_2\}, C)$

[152] They are fully faithful for $n = 2$ and faithful for $n = 1$.

the I-th component of the corresponding morphism $\psi : A_1^\ell \otimes A_2^\ell \to C^\ell$ sends $m_1 \otimes m_2$, where $m_1 \in A^\ell_{1X^I}, m_2 \in A^\ell_{2X^I}$, to the pull-back of $\varphi_1(m_1) \boxtimes \varphi_2(m_2)$[153] by the diagonal map $X^I \hookrightarrow X^I \times X^I$. The details are left to the reader. □

Let us show that the functor $A \mapsto A^{as}$ (see 3.4.2) commutes with tensor products:

3.5.17 Lemma. There is a canonical isomorphism $\hat{\otimes} A_\alpha^{as} \xrightarrow{\sim} (\otimes A_\alpha)^{as}$ of topological associative algebras on X equipped with a connection.

Proof. Recall that A^{as} equals the projective limit of $\mathcal{O}_X$-modules $\Delta^* A^\ell_\xi$ where $A_\xi \in \Xi^{as}(A)$, i.e., $A_\xi \subset j_* j^*(A \boxtimes \mathcal{O}_X)$ is a chiral subalgebra along the first variable equal to $A \boxtimes \mathcal{O}_X$ on U (see 3.4.2). It is clear that any subalgebra $(\otimes A_\alpha)_\xi$ contains a subalgebra of type $\otimes(A_{\alpha\xi_\alpha})$.[154] This yields a canonical isomorphism of topological $\mathcal{O}_X$-modules $\hat{\otimes} A_\alpha^{as} := \varprojlim \otimes \Delta^* A^\ell_{\alpha\xi_\alpha} \xrightarrow{\sim} (\otimes A_\alpha)^{as}$. It follows from definitions that our isomorphism is compatible with products and connections. □

3.5.18. A *Hopf chiral algebra* is a chiral algebra A equipped with a coassociative morphism of chiral algebras $\delta : A \to A \otimes A$. If A is Hopf algebra then, according to 3.5.17, A^{as} is a topological Hopf algebra equipped with a connection.

Remark. Contrary to the usual situation there is no natural notion of action of a Hopf chiral algebra on another chiral algebra.

3.5.19. Let us describe modules over chiral algebras in factorization terms. Below for a finite set I we set $\widetilde{I} := I \sqcup \cdot$, so $X^{\widetilde{I}} = X \times X^I$.

Let B be any factorization algebra. A *factorization B-module* is a pair $(M, \tilde{\kappa})$ where:

(i) M is a rule that assigns to every finite set I a left $\mathcal{D}$-module $M^\ell_{X^{\widetilde{I}}}$ on $X \times X^I$, and to every $\pi : \widetilde{J} \twoheadrightarrow \widetilde{I}$ preserving $\cdot$'s an identification $\nu^{(\pi)} : \Delta^{(\pi)*} M^\ell_{X^{\widetilde{J}}} \xrightarrow{\sim} M^\ell_{X^{\widetilde{I}}}$. We demand that $\nu^{(\pi)}$'s are compatible with composition of π's, and $M^\ell_{X^{\widetilde{I}}}$ is flat along the diagonals.

(ii) Consider a surjection $\widetilde{J} \twoheadrightarrow \widetilde{I}$ preserving $\cdot$'s. Then $\widetilde{J}$ is disjoint union of the preimage subsets $\widetilde{J}_\cdot$ and J_i, $i \in I$. Our $\tilde{\kappa}$ assigns to such surjection an isomorphism of $\mathcal{D}_{U[\widetilde{J}/\widetilde{I}]}$-

[153] To see that this section belongs to $C^\ell_{X^I \times X^I}$ use 3.5.5 and the fact that $\varphi_{1,2}$ mutually commute.

[154] Take for $A_{\alpha\xi_\alpha}$ the preimage of $(\otimes A_\alpha)_\xi$ by the morphism $A_\alpha \to \otimes A_\alpha$.

modules[155]

$$\tilde{\kappa}_{[\tilde{J}/\tilde{I}]} : j^{[\tilde{J}/\tilde{I}]*}(M^{\ell}_{X^{\tilde{J}}} \boxtimes (\boxtimes_I B_{X^{J_i}})) \xrightarrow{\sim} j^{[\tilde{J}/\tilde{I}]*} B_{X^J}. \tag{3.5.19.1}$$

We demand that $\tilde{\kappa}$'s are mutually compatible: for $\tilde{K} \twoheadrightarrow \tilde{J}$ the isomorphism $\tilde{\kappa}_{[\tilde{K}/\tilde{J}]}$ coincides with the composition $\tilde{\kappa}_{[\tilde{K}/\tilde{I}]}(\boxtimes \tilde{\kappa}_{[\tilde{K}_i/\tilde{J}_i]})$.

We denote the category of factorization B-modules by $\mathcal{M}^{\ell}(X, B)$. This is an abelian k-category.

§6 Operator product expansions

In this section we give a third definition of chiral algebras. It shows that our chiral algebras are related to their namesake from mathematical physics, and (in the universal setting) are essentially the same as vertex operator algebras from references cited in 0.0. The results of this section will not be used in the rest of the work.

3.6.1. Let us begin with some preliminary considerations. For a smooth variety P a *$\mathcal{D}_P$-sheaf* is any (not necessary quasicoherent) sheaf of left $\mathcal{D}_P$-modules. We denote the category of $\mathcal{D}_P$-sheaves by $\bar{\mathcal{M}}^{\ell}(P)$, so $\mathcal{M}^{\ell}(P)$ is a full subcategory of $\bar{\mathcal{M}}^{\ell}(P)$. The $\mathcal{O}_P$-tensor product defines on $\bar{\mathcal{M}}^{\ell}(P)$ the structure of tensor category. Let $i : Z \hookrightarrow P$ be a closed smooth subvariety, $\mathcal{I} \subset \mathcal{O}_P$ the corresponding ideal. If F is a $\mathcal{D}_P$-sheaf then $i^*F := i^{\cdot}(F/\mathcal{I}F)$ is a $\mathcal{D}_Z$-sheaf in the usual manner. The functor i^* commutes with tensor products.

3.6.2 Lemma. (i) The functor $i^* : \bar{\mathcal{M}}^{\ell}(P) \to \bar{\mathcal{M}}^{\ell}(Z)$ admits a right adjoint functor $i^{\wedge}_* : \bar{\mathcal{M}}^{\ell}(Z) \to \bar{\mathcal{M}}^{\ell}(P)$. the functor $i^{\wedge}_*$ is exact and fully faithful. A $\mathcal{D}_P$-sheaf belongs to its image if and only if $F = \varprojlim F/\mathcal{I}^n F$.

(ii) For a $\mathcal{D}_Z$-sheaf G set $i_*G := (i_*\omega_Z)^{\ell} \otimes i^{\wedge}_* G$.[156] The functor $i_* : \bar{\mathcal{M}}^{\ell}(Z) \to \bar{\mathcal{M}}^{\ell}(P)$ is exact and fully faithful. A $\mathcal{D}_P$-sheaf belongs to its image if and only if every its local section is killed by some power of $\mathcal{I}$.

[155] See 3.5.4 for notation.

[156] Here $(i_*\omega_Z)^{\ell} \in \mathcal{M}^{\ell}(P)$, see 2.1.1, 2.1.3.

(iii) The above i_* coincides on $\mathcal{M}^\ell(Z)$ with the direct image functor from 2.1.3: for $N \in \mathcal{M}(Z)$ one has $i_*(N^\ell) = (i_*N)^\ell$.

Proof. Left to the reader. □

Remark. For $A, B \in \bar{\mathcal{M}}^\ell(Z)$ we have a canonical morphism $(i_*^\wedge A) \otimes (i_*^\wedge B) \to i_*^\wedge(A \otimes B)$ (defined by adjunction) and isomorphism $(i_*^\wedge A) \otimes (i_* B) \xrightarrow{\sim} i_*(A \otimes B)$. So if A is a $\mathcal{D}_Z$-algebra then $i_*^\wedge A$ is a $\mathcal{D}_P$-algebra, and if N is an A-module then i_*N is an $i_*^\wedge A$-module.

3.6.3. Assume that Z is a divisor; let $j : U := P \smallsetminus Z \hookrightarrow P$ be the complementary open embedding. For a $\mathcal{D}_Z$-sheaf G set

$$i_*^\sim G := (i_*^\wedge G) \otimes j_*\mathcal{O}_U \in \bar{\mathcal{M}}^\ell(P). \tag{3.6.3.1}$$

It is easy to see that the canonical map $i_*^\wedge G \to i_*^\sim G$ is injective. Since $(i_*\omega_Z)^\ell = j_*\mathcal{O}_U/\mathcal{O}_U$ we get a canonical exact sequence

$$0 \to i_*^\wedge G \to i_*^\sim G \to i_* G \to 0. \tag{3.6.3.2}$$

3.6.4 Lemma. For any $\mathcal{D}_P$-sheaf F the obvious arrows

$$\mathrm{Hom}(F, i_*^\sim G) \longleftarrow \mathrm{Hom}(F \otimes j_*\mathcal{O}_U, i_*^\sim G) \longrightarrow \mathrm{Hom}(F \otimes j_*\mathcal{O}_U, i_* G)$$

are isomorphisms.

Proof. For the left arrow the situation is clear. The inverse to the right arrow sends $\alpha \in \mathrm{Hom}(F \otimes j_*\mathcal{O}_U, i_*G)$ to $\tilde{\alpha} \in \mathrm{Hom}(F \otimes j_*\mathcal{O}_U, i_*^\sim G)$ defined by formula $\tilde{\alpha}(f) = \varprojlim q^n \alpha(q^{-n} f)$. Here $f \in F \otimes j_*\mathcal{O}_U$, $q \in \mathcal{O}_P$ an equation of Z, so $q^n \alpha(q^{-n} f) \in i_*^\sim G / \mathcal{I}^n i_*^\wedge G$. □

Remark. We also have canonical isomorphisms

$$\mathrm{Hom}(F, i_*^\sim G) \xleftarrow{\sim} \varinjlim \mathrm{Hom}(F_\xi, i_*^\wedge G) \xrightarrow{\sim} \varinjlim \mathrm{Hom}(i^* F_\xi, G). \tag{3.6.4.1}$$

where F_ξ runs the set of $\mathcal{D}_P$-subsheaves of $F \otimes j_*\mathcal{O}_U$ such that $F_\xi \cdot j_*\mathcal{O}_U = F \otimes j_*\mathcal{O}_U$. The first arrow assigns to $F_\xi \to i_*^\wedge G$ its tensor product with $j_*\mathcal{O}_U$, the second arrow is the adjunction isomorphism. The inverse map to the first arrow sends $\varphi : F \otimes j_*\mathcal{O}_U \to i_*^\sim G$ to $\varphi_\xi : F_\xi \to i_*^\wedge G$ where $F_\xi := \varphi^{-1}(i_*^\wedge G)$, φ_ξ is the restriction of φ to F_ξ.

Notice that if both F and G are quasicoherent then, as follows from the above lemma, we may assume in (3.6.4.1) that all F_ξ's are quasicoherent.

3.6.5. Let us return to our situation, so X is a curve. For a finite non-empty I and a $\mathcal{D}_X$-sheaf G set $\hat{\Delta}_*^{(I)}G := \Delta^{(I)\hat{}}_*G$ and

$$\tilde{\Delta}_*^{(I)}G := (\hat{\Delta}_*^{(I)}G) \otimes j_*^{(I)}\mathcal{O}_{U^{(I)}} \in \bar{\mathcal{M}}(X^I). \tag{3.6.5.1}$$

The obvious morphism $\hat{\Delta}_*^{(I)}G \to \tilde{\Delta}_*^{(I)}G$ is injective. Note that $\hat{\Delta}_*^{(I)}G$ has the following explicit description. For $i \in I$ consider a $\mathcal{D}_{X^I}$-sheaf $pr_i^*G := G \boxtimes \mathcal{O}_X^{\boxtimes I \smallsetminus \{i\}}$. Then $\Delta^{(I)\flat}pr_i^*G = G$, hence $\hat{\Delta}_*^{(I)}G = \varprojlim pr_i^*G/\mathcal{I}_\Delta^n pr_i^*G$ where $\mathcal{I}_\Delta \subset \mathcal{O}_{X^I}$ is the ideal of the diagonal $X \subset X^I$.

More generally, consider a surjective map $I \twoheadrightarrow S$ of finite sets. Let $\Delta^{(I/S)} : X^S \hookrightarrow X^I$ be the diagonal embedding, and $j^{(I/S)} : U^{(I/S)} \hookrightarrow X^I$ be the complement to the diagonals containing X^S. For a $\mathcal{D}_{X^S}$-sheaf F set

$$\tilde{\Delta}_*^{(I/S)}F := (\hat{\Delta}_*^{(I/S)}F) \otimes j_*^{(I/S)}\mathcal{O}_{U^{(I/S)}} \in \bar{\mathcal{M}}(X^I). \tag{3.6.5.2}$$

Note that if F is a $j_*^{(S)}\mathcal{O}_{U^{(S)}}$-module then $\tilde{\Delta}_*^{(I/S)}F$ is a $j_*^{(I)}\mathcal{O}_{U^{(I)}}$-module.

3.6.6. Consider the Cousin complex for ω_{X^I} with respect to the diagonal stratification (see the proof of 3.1.4). Tensoring it by $(\hat{\Delta}_*^{(I)}G)\omega_{X^I}^{-1}$ we get the complex $C_I(G)$ of $\mathcal{D}_{X^I}$-sheaves with terms $C_I(G)^m := \underset{T \in Q(I,|I|-m)}{\oplus} \Delta_*^{(I/T)}\tilde{\Delta}_*^{(T)}G$ (in particular, $C_I(G)^0 = \tilde{\Delta}_*^{(I)}G$) which is a resolution of $\hat{\Delta}_*^{(I)}G$.

3.6.7 Lemma. Assume that a $\mathcal{D}_{X^I}$-sheaf N is a $j_*^{(I)}\mathcal{O}_{U^{(I)}}$-module. Then the sequence

$$0 \to \mathrm{Hom}(N, C_I(G)^0) \to \mathrm{Hom}(N, C_I(G)^1) \to \mathrm{Hom}(N, C_I(G)^2)$$

is exact.

Proof. The statement is X-local, so let us choose an equation $q = 0$ of the diagonal divizor. Then $\tilde{\Delta}_*^{(I)}G = \varprojlim \tilde{\Delta}_*^{(I)}G/q^n\hat{\Delta}_*^{(I)}G$, and for any n the sequence $0 \to \tilde{\Delta}_*^{(I)}G/q^n\hat{\Delta}_*^{(I)}G \to C_I^0 \xrightarrow{q^{-n}d^0} C_I(G)^1 \xrightarrow{d^1} C_I(G)^2$ is exact. We are done. □

3.6.8. For an I-family of $\mathcal{D}_X$-sheaves set

$$O_I(\{F_i\}, G) := \mathrm{Hom}(\boxtimes F_i, \tilde{\Delta}_*^{(I)}G) = \mathrm{Hom}((\boxtimes F_i) \otimes j_*^{(I)}\mathcal{O}_{U^{(I)}}, \tilde{\Delta}_*^{(I)}G). \tag{3.6.8.1}$$

This is the vector space of *ope I-operations* where ope stands for "operator product expansion". they have local nature with respect to the étale topology of X.

The composition of ope operations need not be an ope operation: it belongs to a larger vector space. Namely, for a surjective map $J \twoheadrightarrow I$ set

$$\tilde{\Delta}_*^{(J,I)}G := \tilde{\Delta}_*^{(J/I)}\tilde{\Delta}_*^{(I)}G. \tag{3.6.8.2}$$

since $\hat{\Delta}_*^{(J)} = \hat{\Delta}_*^{(J/I)}\hat{\Delta}_*^{(I)}$ we have a canonical embedding $\tilde{\Delta}_*^{(J)}G \hookrightarrow \tilde{\Delta}_*^{(J,I)}G$.

Now for a J-family of $\mathcal{D}_X$-sheaves the *composition map*

$$O_I(\{F_i\}, G) \otimes (\underset{I}{\otimes} O_{J_i}(\{E_j\}, F_i)) \longrightarrow \mathrm{Hom}(\boxtimes E_j, \tilde{\Delta}_*^{(J,I)}G) \tag{3.6.8.3}$$

sends $\gamma \otimes (\otimes \delta_i)$ to $\gamma(\delta_i)$ defined as composition

$$\underset{J}{\boxtimes} E_j \xrightarrow{\boxtimes \delta_i} \underset{I}{\boxtimes} \tilde{\Delta}_*^{(J_i)}F_i \longrightarrow \hat{\Delta}_*^{(J/I)}(\boxtimes F_i) \otimes j_*^{(J)}\mathcal{O}_{U^{(J)}} \longrightarrow \tilde{\Delta}_*^{(J,I)}G.$$

Here the second arrow comes from the obvious morphism $\boxtimes \hat{\Delta}_*^{(J_i)}F_i \to \hat{\Delta}_*^{(J/I)}(\boxtimes F_i)$, and the third one is $\hat{\Delta}_*^{(J/I)}(\gamma)$ tensored by identity morphism for $j_*^{(J)}\mathcal{O}_{U^{(J)}}$.

We say that ope operations $(\gamma, \{\delta_i\})$ *compose nicely* if $\gamma(\delta_i)$ takes values in $\tilde{\Delta}_*^{(J)}G \subset \tilde{\Delta}_*^{(J/I)}G$. Then $\gamma(\delta_i) \in O_J(\{E_j\}, G)$.

3.6.9. The composition of ope operations is associative in the following sense. Let $K \twoheadrightarrow J$ be another surjection. Set $\tilde{\Delta}_*^{(K,J,I)}G := \tilde{\Delta}_*^{(K,J)}\tilde{\Delta}_*^{(J,I)}G \in \bar{\mathcal{M}}^\ell(X^K)$. There is a commutative diagram of natural embeddings $\tilde{\Delta}_*^{(K)}G \subset \tilde{\Delta}_*^{(K,I)}G, \tilde{\Delta}_*^{(K,J)}G \subset \tilde{\Delta}_*^{(K,J,I)}G$.

Now for a K-family $\{C_k\}$ of $\mathcal{D}_X$-sheaves and operations $\varepsilon \in O_{K_j}(\{C_k\}, E_j)$ the double composition morphisms $(\gamma(\delta_i))(\varepsilon_j), \gamma(\delta_i(\varepsilon_j)) \in \mathrm{Hom}(\boxtimes C_k, \tilde{\Delta}_*^{(K,J,I)}G)$ are defined, respectively, as compositions

$$\boxtimes C_k \xrightarrow{\boxtimes \varepsilon_j} \boxtimes \tilde{\Delta}_*^{(K_j)}E_j \longrightarrow \tilde{\Delta}_*^{(K/J)}(\boxtimes E_j) \xrightarrow{\tilde{\Delta}_*^{(K/J)}(\gamma(\delta_i))} \tilde{\Delta}_*^{(K,J,I)}G,$$

$$\boxtimes C_k \xrightarrow{\boxtimes \delta_i(\varepsilon_j)} \boxtimes \tilde{\Delta}_*^{(K_i,J_i)}F_i \longrightarrow \tilde{\Delta}_*^{(K/J)}\tilde{\Delta}_*^{(J/I)}(\boxtimes F_i) \longrightarrow \tilde{\Delta}_*^{(K,J,I)}G.$$

The associativity property says that they coincide.

3.6.10. Let $\circ \in O_2(\{G,G\},G)$ be a binary ope operation, $a \boxtimes b \mapsto a \circ b \in \tilde{\Delta}_* G$[157]. We say that $\circ$ is *commutative* if it is fixed by an obvious "transposition of coordinates" symmetry of $O_2(\{G,G\},G)$, and $\circ$ is *associative* if both $(\circ, \{\circ, id_G\})$, $(\circ, \{id_G, \circ\})$ compose nicely and the composition operations $\in O_3(\{G,G,G\},G)$ coincide.

We say that $\circ$ is an *ope algebra product*, or $(G,\circ)$ is an *ope algebra*, if $\circ$ is both commutative and associative. Then the triple product $\circ_3 := \circ(\{\circ, id_G\})$ is fixed by the action of symmetric group of 3 variables.

A *unit* in ope algebra G is a horizontal section $1 \in G$ such that for every $a \in G$ one has $a \circ 1 = pr_1^* a \in \varprojlim pr_1^* G/\mathcal{I}_\Delta pr_1^* G = \hat{Delta}_* G \subset \tilde{\Delta}_* G$ (see 3.6.5). A unit is unique, if it exists.

3.6.11. Let L_1, L_2, M be right $\mathcal{D}_X$-modules. Then 3.6.2(iii), 3.6.4 yield a canonical isomorphism

$$P_2^{ch}(\{L_1,L_2\},M) \otimes \lambda_{\{1,2\}} \xrightarrow{\sim} O_2(\{L_1^\ell, L_2^\ell\}, M^\ell). \tag{3.6.11.1}$$

Here we use the identification $\omega_X \boxtimes \Omega_X = \omega_{X\times X} \otimes \lambda_{\{1,2\}}$, see 2.2.2.

Now we may state the main result of this section:

3.6.12 Proposition. For any $\mathcal{D}_X$-module A the above isomorphism

$$P_2^{ch}(\{A,A\},A) \otimes \lambda_{\{1,2\}} \xrightarrow{\sim} O_2(\{A^\ell, A^\ell\}, A^\ell), \qquad \mu \mapsto \circ_\mu,$$

identifies the set of chiral Lie brackets with the set of ope algebra products. A horizontal section $1 \in A^\ell$ is a unit for μ if and only if it is a unit for $\circ_\mu$.

Proof. We will prove the first statement; the second one is left to the reader.

Since (3.6.11.1) commutes with transposition of coordinates, skew-commutative μ correspond to commutative $\circ_\mu$. Take such μ. Let us show that the Jacobi identity for μ amounts to associativity of $\circ = \circ_\mu$.

Set $R := j_*^{(I)} j^{(I)*}(A^{\ell \boxtimes I})$. By 3.6.4 for every $S \in Q(I, |I|-1)$ we have isomorphisms

$$\mathrm{Hom}(A^{\ell\boxtimes I}, \tilde{\Delta}_*^{(I,S)} A^\ell) \xleftarrow{\sim} \mathrm{Hom}(R, \tilde{\Delta}_*^{(I/S)} \tilde{\Delta}_*^{(S)} A^\ell) \xrightarrow{\sim} \mathrm{Hom}(R, \Delta_*^{(I/S)} \tilde{\Delta}_*^{(S)} A^\ell).$$

[157] Here $\Delta := \Delta^{(\{1,2\})}$.

Since $\mathrm{Hom}(R, \tilde{\Delta}_*^{(I)} A^\ell = \mathrm{Hom}(A^{\ell\boxtimes I}, \tilde{\Delta}_*^{(I)} A^\ell)$, the exact sequence from 3.6.7 looks as

$$0 \to \mathrm{Hom}(A^{\ell\boxtimes I}, \tilde{\Delta}_*^{(I)} A^\ell) \xrightarrow{d^0} \bigoplus_{S\in Q(I,|I|-1)} \mathrm{Hom}(A^{\ell\boxtimes I}, \tilde{\Delta}_*^{(I,S)} A^\ell)$$
$$\xrightarrow{d^1} \bigoplus_{S\in Q(I,|I|-2)} \mathrm{Hom}(R, \Delta_*^{(I/S)} \tilde{\Delta}_*^{(S)} A^\ell).$$

The symmetric group of I indices acts on it in the obvious way.

Now assume that $I = \{1,2,3\}$. The third vector space in the above exact sequence is $\mathrm{Hom}(j_*^{(3)} j^{(3)*}(A^{\ell\boxtimes 3}), \Delta_*^{(3)} A^\ell) = P_3^{ch}(\{A,A,A\}, A) \otimes \lambda_{\{1,2,3\}}$ (use 3.6.2(iii) and the identification $\omega_X^{\boxtimes 3} = \omega_{X^3} \otimes \lambda_{\{1,2,3\}}$). Our $\circ$ is commutative, so for every $S \in Q(I,2)$ we have the corresponding iterated product $\bar{\circ}_S \in \mathrm{Hom}(A^{\ell\boxtimes 3}, \tilde{\Delta}_*^{(I,S)} A^\ell)$. For example, if S identifies $1,2 \in I$ then $\bar{\circ}_S = \circ(\circ, id)$. The element $\bar{\circ}_3 := \Sigma \bar{\circ}_S$ of the middle term of the above exact sequence is invariant with respect to the action of the symmetric group. As follows from the construction, one has $d^1(\circ(\circ, id)) = \mu(\mu, id_A)$. Therefore $d^1(\bar{\circ}_3)$ is exactly the sum of terms of the Jacobi identity, so $d^1(\bar{\circ}_3) = 0$ if and only if μ is a Lie bracket. On the other hand, by the very definition, $\circ$ is associative if and only if $\bar{\circ}_3$ lies in the image of d^0. Since our sequence is exact, we are done. □

3.6.13 Lemma. Let $(G, \circ)$ be an ope algebra. then for any finite non-empty I one has a canonical *I-fold ope product* operation $\circ_I \in O_I(\{G\}, G)$. These operations are uniquely determined by the following properties:

(i) If $|I| = 1$ then $\circ_I = id_G$; if $|I| = 2$ then $\circ_I = \circ$.

(ii) For every $J \twoheadrightarrow I$ the operations $(\circ_I, \{\circ_{J_i}\})$ compose nicely; the composition is $\circ_J$.

Proof. We define $\circ_I$ by induction. We know $\circ_I$ if $|I| \leq 3$. Now let I be a finite set of order $n > 3$, and assume that we know $\circ_{I'}$ for $|I'| < n$ so that (i), (ii) hold. Consider the exact sequence from the proof of 3.6.12. For $S \in Q(I, n-1)$ set $\bar{\circ}_S := \circ_S(\circ_{\{a,b\}}, id_{G_i})_{i\neq a,b} \in \mathrm{Hom}(G^{\boxtimes I}, \tilde{\Delta}^{(I,S)} G)$. Here $a, b \in I$ are the two distinct S-equivalent elements. Set $\bar{\circ}_I := \Sigma \bar{\circ}_S \in \bigoplus_{S\in Q(I,n-1)} \mathrm{Hom}(G^{\boxtimes I}, \tilde{\Delta}^{(I,S)} G)$. Associativity of composition 3.6.9 and the existence of $\circ_3$ imply that $d^1(\bar{\circ}_I) = 0$. Hence $\bar{\circ}_I = d^0(\circ_I)$ for certain $\circ_I$ uniquely determined by this condition. Property (ii) follows from associativity of composition. □

3.6.14. Let us translate some of the definitions from 3.3 in the ope language. Let (A, μ) be a chiral algebra, and $(A^\ell, \circ)$ the corresponding ope algebra.

By definition, the Lie* bracket $[\ \ ]_\mu : A \boxtimes A \to \Delta_* A$ coincides with the composition $A^\ell \boxtimes A^\ell \xrightarrow{\circ} \tilde{\Delta}_* A^\ell \to \Delta_* A^\ell$ tensored by $id_{\omega_{X^2}}$, hence $[\ \ ]_\mu$ is the *singular part* of $\circ$. Therefore our chiral algebra is commutative if and only if $\circ$ takes values in $\hat{\Delta}_* A^\ell \subset \tilde{\Delta}_* A^\ell$. By adjunction property of $\hat{\Delta}_*$ (see 3.6.2(i)) one may rewrite such $\circ$ as a morphism $\cdot : \check{\Delta}(A^\ell \boxtimes A^\ell) = A^\ell \otimes A^\ell \to A^\ell$. Associativity and commutativity of $\circ$ amount to similar properties of $\cdot$, so we get another proof of 3.3.2.1, 3.3.3.1.

3.6.15. If $(G, \circ)$ is an ope algebra then a *G-module* is a $\mathcal{D}_X$-sheaf N together with an ope pairing $\circ_N \in O_2(\{G, N\}, N)$ such that both $\circ_N(\circ, id_N)$ and $\circ_N(id_G, \circ_N)$ compose nicely and the compositions $\in O_3(\{G, G, N\}, N)$ coincide. If G has unit, then one defines a unital G-module in the obvious manner.

Let A be a chiral algebra as above, M an A-module, so we have the chiral action $\mu_M \in P_2^{ch}(\{A, M\}, M)$. Let $\circ_M \in O_2(\{A^\ell, M^\ell\}, M^\ell)$ be the corresponding ope pairing (see 3.6.11). This is an $(A^\ell, \circ)$-module structure on M. This identifies the category of (A, μ)-modules with that of (quasicoherent) $(A^\ell, \circ)$-modules. This follows directly from 3.6.12 (use 3.3.5).

§7 Chiral enveloping algebras

Here is the key result of this section:

3.7.1 Theorem. The forgetful functor $\mathcal{CA}(X) \to \mathcal{L}ie^*(X)$, $A \mapsto A^{(*)}$, admits a left adjoint functor $U : \mathcal{L}ie^*(X) \to \mathcal{CA}(X)$.

For a Lie* algebra L we call $U(L)$ *the chiral enveloping algebra*, or simply *the chiral envelope* of L.

We give two different proofs of the above theorem – in the chiral (3.7.2–3.7.4) and factorization (3.7.5–3.7.11) settings respectively. The factorization picture has a definite advantage that it automatically yields immediately the Poincaré-Birkhoff-Witt theorem

3.7.14 (as well as computation of chiral homology of enveloping algebras, see 3.11.?).[158]

In the chiral setting we prove a more general result. To formulate it we need the following definition:

Definition. A k-operad $\mathcal{B}$ is *of Jacobi type* if it is generated by $\mathcal{B}_1$ and $\mathcal{B}_2$, and for every $\alpha, \alpha' \in \mathcal{B}_2$ there exist $\beta, \beta', \gamma, \gamma' \in \mathcal{B}_2$ such that $\alpha(x_1, \alpha'(x_2, x_3)) = \beta(x_3, \beta'(x_1, x_2)) + \gamma(x_2, \gamma'(x_3, x_1))$.

For example, the operads $\mathcal{L}ie$, $\mathcal{P}oiss$, $\mathcal{C}om$, $\mathcal{A}ss$ are of Jacobi type.

3.7.2 Theorem. For an operad $\mathcal{B}$ of Jacobi type the functor $\beta^{\mathcal{B}}$ of (3.3.1.1) admits a left adjoint functor $U_{\mathcal{B}} : \mathcal{B}^*(X) \to \mathcal{B}^{ch}(X)$.

Composing $U_{\mathcal{L}ie}$ with the "adding of unit" functor (see 3.3.3) we get the functor U of 3.7.1.

Proof of 3.7.2. Here is the idea. If the chiral pseudo-tensor structure were representable (see 1.1.3) then 3.7.2 would be immediate for arbitrary $\mathcal{B}$. Indeed, representability implies that for every L the free $\mathcal{B}^{ch}$ algebra generated by L is well-defined. So, if L is a $\mathcal{B}^*$ algebra, then $U_{\mathcal{B}}(L)$ is a quotient of the free $\mathcal{B}^{ch}$ algebra modulo the obvious relations. Now the *binary* chiral pseudo-tensor product is representable up to the non-representability of the $*$ pseudo-tensor product. This shows that the *quadratic* part of $U_{\mathcal{B}}(L)$ is well-defined. To get control of the whole $U_{\mathcal{B}}(L)$ one needs the Jacobi property of $\mathcal{B}$.

Let us turn to the actual proof. The first step (3.7.3 below) is to embed our pseudo-tensor category in a representable (actually, a tensor) one:

3.7.3. (i) A *right $\mathcal{D}$-module M on $X^{\mathcal{S}}$* is a rule that assigns to $I \in \mathcal{S}$ a right $\mathcal{D}$-module M_{X^I} on X^I and to $\pi : J \twoheadrightarrow I$ a morphism of $\mathcal{D}$-modules $\theta^{(\pi)} : \Delta_*^{(\pi)} M_{X^I} \to M_{X^J}$. We demand that $\theta^{(\pi)}$'s are compatible with composition of π's[159] and $\theta^{(id_I)} = id_{M_{X^I}}$.

These objects form an abelian k-category $\mathcal{M}(X^{\mathcal{S}})$. There is an exact fully faithful embedding $\mathcal{M}(X) \hookrightarrow \mathcal{M}(X^{\mathcal{S}})$ which identifies a $\mathcal{D}$-module M on X with the one on $X^{\mathcal{S}}$ given by $M_{X^I} := \Delta_*^{(I)} M$, $\theta^{(\pi)} = id_{\Delta_*^{(J)} M}$. This embedding is left adjoint to the projection functor $\mathcal{M}(X^{\mathcal{S}}) \to \mathcal{M}(X)$, $M \mapsto M_X$.

[158] In the chiral setting the proof of PBW takes more time, see 3.7.26–3.7.??.

[159] i.e., $\theta^{(\pi_1 \pi_2)} = \theta^{(\pi_2)} \Delta_*^{(\pi_2)}(\theta^{(\pi_1)})$.

Let $\mathcal{M}(X^{\mathcal{S}})_\Delta \subset \mathcal{M}(X^{\mathcal{S}})$ be the full subcategory of those M that every M_{X^I} is supported on the diagonal $X \subset X^I$; it is closed under subquotients and extensions. $\mathcal{M}(X^{\mathcal{S}})_\Delta$ is equivalent to the category $\mathcal{M}(X)^{\mathcal{S}}$ of all functors $\mathcal{S}^\circ \to \mathcal{M}(X)$ (one identifies $M \in \mathcal{M}(X^{\mathcal{S}})_\Delta$ with the functor $I \mapsto \Delta^{(I)!}M_{X^I}$). Notice that $\mathcal{M}(X) \subset \mathcal{M}(X^{\mathcal{S}})_\Delta$, and this embedding admits a left adjoint functor $\epsilon : \mathcal{M}(X^{\mathcal{S}})_\Delta \to \mathcal{M}(X)$, $M \mapsto \varinjlim \Delta^{(I)!}M_{X^I}$.

Remark. For $n \geq 1$ let $\mathcal{S}_n \subset \mathcal{S}$ be the subcategory of sets of order $\leq n$. The above category has an obvious "truncated" version $\mathcal{M}(X^{\mathcal{S}_n})$. The "restriction" functor $\mathcal{M}(X^{\mathcal{S}}) \xrightarrow{\pi_n} \mathcal{M}(X^{\mathcal{S}_n})$ admits a left adjoint $a_n : \mathcal{M}(X^{\mathcal{S}_n}) \to \mathcal{M}(X^{\mathcal{S}})$, $a_n(N)_{X^I} = \varinjlim \Delta_*^{(I/S)} N_{X^S}$ where the inductive limit is taken over the ordered set $Q(I, \leq n)$. Since $\pi_n a_n$ is the identity functor we see that π_n identifies $\mathcal{M}(X^{\mathcal{S}_n})$ with the quotient category of $\mathcal{M}(X^{\mathcal{S}})$ modulo the full subcategory $\mathcal{M}(X^{\geq n+1})$ of such M's that $\pi_n M = 0$, i.e., $M_{X^I} = 0$ if $|I| \leq n$. In particular, $\mathrm{gr}_n\mathcal{M}(X^{\mathcal{S}}) := \mathcal{M}(X^{\geq n})/\mathcal{M}(X^{\geq n+1})$ equals the category of Σ_n-equivariant $\mathcal{D}$-modules on X^n.

(ii) $\mathcal{M}(X^{\mathcal{S}})$ carries two tensor products $\otimes^*$ and $\otimes^{ch}$ defined as follows. Let M_α, $\alpha \in A$ be a finite non-empty family of objects of $\mathcal{M}(X^{\mathcal{S}})$. One has

$$(\otimes_A^* M_\alpha)_{X^I} := \bigoplus_{I \twoheadrightarrow A} \underset{A}{\boxtimes} (M_{\alpha X^{I_\alpha}}), \tag{3.7.3.1}$$

the arrows $\theta^{(\pi)}$ are the obvious ones. For example, if $M_\alpha \in \mathcal{M}(X)$ then $(\otimes^* M_\alpha)_{X^I} = \bigoplus_{I \twoheadrightarrow A} \Delta_*^{(I/A)} \underset{A}{\boxtimes} M_\alpha$; in particular, $(\otimes_A^* M_\alpha)_{X^A} = \bigoplus_{\nu \in \mathrm{Aut}A} \nu^*(\boxtimes M_\alpha)$. Similarly, set

$$(\otimes_A^{ch} M_\alpha)_{X^I} := \bigoplus_{I \twoheadrightarrow A} j_*^{[J/A]} j^{[J/A]*} \boxtimes M_{\alpha X^{I_\alpha}}, \tag{3.7.3.2}$$

the arrows $\theta^{(\pi)}$ are obvious ones. Here $j^{[J/A]}$ was defined in 3.5.4.

Our tensor products are subject to obvious "associativity and commutativity constraints", so they define on $\mathcal{M}(X^{\mathcal{S}})$ tensor category structures which we denote by $\mathcal{M}(X^{\mathcal{S}})^*$, $\mathcal{M}(X^{\mathcal{S}})^{ch}$. We have an obvious canonical morphism

$$\otimes^* M_\alpha \to \otimes^{ch} M_\alpha, \tag{3.7.3.3}$$

so the identity functor for $\mathcal{M}(X^{\mathcal{S}})$ extends to a pseudo-tensor functor

$$\beta^{\mathcal{S}} : \mathcal{M}(X^{\mathcal{S}})^{ch} \to \mathcal{M}(X^{\mathcal{S}})^*. \tag{3.7.3.4}$$

(iii) The embedding $\mathcal{M}(X) \hookrightarrow \mathcal{M}(X^{\mathbb{S}})$ extends canonically to fully faithful pseudo-tensor embeddings

$$\mathcal{M}(X)^* \hookrightarrow \mathcal{M}(X^{\mathbb{S}})^*, \quad \mathcal{M}(X)^{ch} \hookrightarrow \mathcal{M}(X^{\mathbb{S}})^{ch}. \tag{3.7.3.5}$$

Namely, for $M_\alpha, N \in \mathcal{M}(X)$ the identification $\mathrm{Hom}(\otimes^*_A M_\alpha, N) \xrightarrow{\sim} P^*_A(\{M_\alpha, N)$ sends a morphism $\varphi : \otimes^*_A M_\alpha \to N$ to the restriction of φ_{X^A} to $\boxtimes\limits_A M_\alpha \subset (\otimes^*_A M_\alpha)_{X^A}$. The identification $\mathrm{Hom}(\otimes^*_A M_\alpha, N) \xrightarrow{\sim} P^*_A(\{M_\alpha, N)$ is defined in the similar way. Notice that the restriction of $\beta_{\mathbb{S}}$ to $\mathcal{M}(X)^*$ is the pseudo-tensor functor β from (3.2.1.2).

The functor $\epsilon : \mathcal{M}(X^{\mathbb{S}})_\Delta \to \mathcal{M}(X)$ extends in the obvious way to pseudo-tensor functors

$$\mathcal{M}(X^{\mathbb{S}})^*_\Delta \to \mathcal{M}(X)^*, \quad \mathcal{M}(X^{\mathbb{S}})^{ch}_\Delta \to \mathcal{M}(X)^{ch} \tag{3.7.3.6}$$

which are left adjoint to the pseudo-tensor functors from (3.7.3.5).

(iv) For any k-operad $\mathcal{B}$ denote by $\mathcal{B}^*(X^{\mathbb{S}})$, $\mathcal{B}^{ch}(X^{\mathbb{S}})$ the categories of $\mathcal{B}$ algebras in $\mathcal{M}(X^{\mathbb{S}})^*$, $\mathcal{M}(X^{\mathbb{S}})^{ch}$ (we call them $\mathcal{B}^*$ and $\mathcal{B}^{ch}$ algebras on $X^{\mathbb{S}}$). Notice that by (3.7.3.4) every $\mathcal{B}$ algebra in $\mathcal{M}(X^{\mathbb{S}})^{ch}$ is automatically a $\mathcal{B}$ algebra in $\mathcal{M}(X^{\mathbb{S}})^*$, i.e., the pseudo-tensor functor $\beta^{\mathbb{S}}$ yields a functor $\mathcal{B}^{ch}(X^{\mathbb{S}}) \to \mathcal{B}^*(X^{\mathbb{S}})$. This functor admits a left adjoint functor

$$U^{\mathbb{S}}_{\mathcal{B}} : \mathcal{B}^*(X^{\mathbb{S}}) \to \mathcal{B}^{ch}(X^{\mathbb{S}}). \tag{3.7.3.7}$$

Indeed, for a $\mathcal{B}^*$ algebra L the corresponding $\mathcal{B}^{ch}$ algebra $U^{\mathbb{S}}_{\mathcal{B}}(L)$ is the quotient of the free $\mathcal{B}^{ch}$ algebra generated by L modulo the relations needed to assure that the canonical morphism $i^{\mathbb{S}} : L \to U^{\mathbb{S}}_{\mathcal{B}}(L)$ is a morphism of $\mathcal{B}^*$ algebras. Precisely, let Φ^*, Φ^{ch} be the free $\mathcal{B}^*$ and $\mathcal{B}^{ch}$ algebras generated by L. Since Φ^{ch} is a $\mathcal{B}^*$ algebra the canonical morphism $i^{ch} : L \to \Phi^{ch}$ extends to a morphism of $\mathcal{B}^*$ algebras $\alpha : \Phi^* \to \Phi^{ch}$. Since L is a $\mathcal{B}^*$ algebra the identity morphism id_L extends to a morphism of $\mathcal{B}^*$ algebras $\beta : \Phi^* \to L$. Now $U^{\mathbb{S}}_{\mathcal{B}}(L)$ is the quotient of Φ^{ch} modulo the $\mathcal{B}^{ch}$ ideal $\mathcal{I}$ generated by the image of $\alpha - i^{ch}\beta : \Phi^* \to \Phi^{ch}$.

3.7.4. We return to the proof of 3.7.2. Let L be a $\mathcal{B}^*$ algebra on X. We want to construct the corresponding universal $\mathcal{B}^{ch}$ algebra $U_{\mathcal{B}}(L)$ on X equipped with a morphism of $\mathcal{B}^*$ algebras $i : L \to U_{\mathcal{B}}(L)$.

Let us consider L as a $\mathcal{B}^*$ algebra on $X^{\mathcal{S}}$, so we have the enveloping algebra $U := U^{\mathcal{S}}_{\mathcal{B}}(L) \in \mathcal{B}^{ch}(X^{\mathcal{S}})$. In the next lemma we will show that $U \in \mathcal{M}(X^{\mathcal{S}})_{\Delta}$. Assuming this, set $U_{\mathcal{B}}(L) := \epsilon(U)$, $i := \epsilon(i^{\mathcal{S}}) : L \to U_{\mathcal{B}}(L)$. By (3.7.3.6) $U_{\mathcal{B}}(L)$ is a $\mathcal{B}$ algebra in $\mathcal{M}(X)^{ch}$ and i is a morphism of $\mathcal{B}^*$ algebras; by the adjunction property of ϵ the pair $(U_{\mathcal{B}}(L), i)$ satisfies the desired universal property. We are done.

It remains to prove the following lemma:

Lemma. If $\mathcal{B}$ is of Jacobi type then $U = U^{\mathcal{S}}_{\mathcal{B}}(L) \in \mathcal{M}(X^{\mathcal{S}})_{\Delta}$.

Proof of Lemma. (a) For a finite set I and $\alpha, \beta \in I$, $\alpha \neq \beta$, let $\mathcal{B}_I^{\alpha\beta} \subset \mathcal{B}_I$ be the image of the composition map $\mathcal{B}_{I/\{\alpha,\beta\}} \otimes \mathcal{B}_{\{\alpha,\beta\}} \to \mathcal{B}_I$. A simple induction shows that $\mathcal{B}$ is of Jacobi type if and only if it satisfies the following property:

For any finite non-empty sets I_1, I_2 the vector space $\mathcal{B}_{I_1 \sqcup I_2}$ is generated by the subspaces $\mathcal{B}^{\alpha\beta}_{I_1 \sqcup I_2}$, $\alpha \in I_1$, $\beta \in I_2$.

(b) Let Φ^{ch} be the free $\mathcal{B}^{ch}$ algebra on $X^{\mathcal{S}}$ generated by L as in 3.7.3(iv). It has a natural gradation $\Phi^{ch} = \underset{n \geq 1}{\oplus} \Phi^{ch}_n$ such that $\Phi^{ch}_n = (L^{\otimes^{ch} I} \otimes \mathcal{B}_I)_{\mathrm{Aut} I}$ where I is a set of order n. This gradation yields an increasing filtration on $U = \Phi^{ch}/\mathfrak{I}$. We are going to show that $\mathrm{gr} U \in \mathcal{M}(X^{\mathcal{S}})_{\Delta}$; this will prove our lemma.

Clearly $\mathrm{Ker}(\Phi^{ch}_n \twoheadrightarrow \mathrm{gr}_n U) \supset N_{\mathrm{Aut} I}$ where $N := \sum_{\alpha \neq \beta \in I} K_{\alpha\beta} \otimes \mathcal{B}_I^{\alpha\beta} \subset L^{\otimes^{ch} I} \otimes \mathcal{B}_I$ and $K_{\alpha\beta}$ is the image of $(L \otimes^* L) \otimes^{ch} L^{\otimes^{ch}(I \smallsetminus \{\alpha,\beta\})}$ in $L^{\otimes^{ch} I}$. So it is enough to show that $M := (L^{\otimes^{ch} I} \otimes \mathcal{B}_I)/N$ belongs to $\mathcal{M}(X^{\mathcal{S}})_{\Delta}$.

Set $U_{\alpha\beta} := \{(x_i) \in X^I : x_i \neq x_j \text{ if } i \neq j, \{i,j\} \neq \{\alpha,\beta\}\}$; let $j_{\alpha\beta} : U_{\alpha\beta} \hookrightarrow X^I$ be the embedding. Set $F := j^{(I)}_* \mathcal{O}_{U^{(I)}} \otimes \mathcal{B}_I$, $G := \sum_{\alpha \neq \beta \in I} j_{\alpha\beta *} \mathcal{O}_{U_{\alpha\beta}} \otimes \mathcal{B}_I^{\alpha\beta} \subset F$, $H := F/G$. By definition, for every $J \in \mathcal{S}$ one has $M_{X^J} = \underset{J \twoheadrightarrow I}{\oplus} \Delta^{(J/I)}_* (L^{\boxtimes I} \otimes H)$. It remains to show that H is supported on the diagonal $X \subset X^I$.

(c) For $I = I_1 \sqcup I_2$ where $I_1, I_2 \neq \emptyset$ set $U_{I_1 I_2} := \{(x_i) \in X^I : x_i \neq x_j \text{ if } i \in I_1, j \in I_2\}$.

On $U_{I_1I_2}$ the sheaves $j_{\alpha\beta *}\mathcal{O}_{U_{\alpha\beta}}$ coincide with $j^{(I)}_*\mathcal{O}_{U^{(I)}}$. So (a) above implies that the restriction of H to $U_{I_1I_2}$ is zero. The union of all $U_{I_1I_2}$'s is the complement to $X \subset X^I$, and we are done. □

3.7.5. Let us return to 3.7.1. We are going to construct the chiral envelope $U(L)$ of a Lie* algebra L explicitely as a factorization algebra (together with the canonical $\mathcal{D}$-module structure). Our problem is X-local, so we can assume that X is affine. To do this we first define in 3.7.6 an auxiliary Lie algebra $L^\natural$ in the tensor category of left $\mathcal{D}$-modules on $\mathcal{R}(X)$. The chiral envelope is constructed in 3.7.7 (it is denoted by V there). The universality property is checked in 3.7.11.

Remark. A slightly unpleasant point of our construction is that the auxiliary Lie algebra $L^\natural$ is a non-local object (while $U(L)$ is of course local). A remedy would be to consider in (3.7.2.1) instead of global cohomology the one of the formal neighbourhood of the diagonal. The price is to deal with non-quasicoherent modules; we prefer not to do this.

In 3.7.6–3.7.11 *we assume that* X *is affine.*

3.7.6. Let I be a finite set. Denote by p_I the projection $X \times X^I \to X^I$, and let $j_I : V = V^{\tilde{}}_I \hookrightarrow X \times X^I$ be the open subset of those $(x,(x_i))$ that $x \in V_{(x_i)} := X \smallsetminus \{x_i\}$. For $L \in \mathcal{M}^r(X)$[160] consider the de Rham complex $DR(L) = \mathcal{C}one(L \otimes \Theta_X \to L)$. Set

$$L^\natural_{X^I} := H^0(p_I j_I).j_I^*(DR(L) \boxtimes \mathcal{O}_{X^I}). \tag{3.7.6.1}$$

This is an $\mathcal{O}_{X^I}$-module with fiber at $(x_i) \in X^I$ equal to $H^0_{DR}(V_{(x_i)}, L) = \Gamma(V_{(x_i)}, h(L))$. The obvious left $p_I^{\cdot}\mathcal{D}^I_X$-module structure on $DR(L) \boxtimes \mathcal{O}_{X^I}$ shows that $L^\natural_{X^I}$ is a left $\mathcal{D}_{X^I}$-module. In other words, $L^{\natural r}_{X^I} = H^0(p_I j_I)_* j_I^*(L \boxtimes \omega_{X^I})$.

Our $L^\natural$ carries a bunch of structures described in (i)-(iii) below that will be used in the construction of the chiral envelope:

(i) For every diagonal embedding $\Delta^{(I/S)} : X^S \hookrightarrow X^I$ we have an obvious identification of left $\mathcal{D}$-modules

$$\Delta^{(I/S)*} L^\natural_{X^I} \xrightarrow{\sim} L^\natural_{X^S}. \tag{3.7.6.2}$$

[160] We do not assume that L is a Lie* algebra at the moment.

So $L^\natural_{X^I}$ form a left $\mathcal{D}$-module on $\mathcal{R}(X)$ (see 3.5.2).

More generally, $L^\natural_{X^I}$ behave nicely with respect to *arbitrary* standard morphisms between X^I's. Namely, if $\pi : J \to I$ is any map of finite sets, then the pull-back of $V_{\tilde{J}}$ by $\Delta^{(\pi)} : X^I \to X^J$ contains $V_{\tilde{I}}$. So we have the restriction map

$$\Delta^{(\pi)*} L^\natural_{X^J} \to L^\natural_{X^I} \tag{3.7.6.3}$$

which coincides with (3.7.6.2) if π is surjective. These maps are injective morphisms of $\mathcal{D}$-modules; they are compatible with composition of π's.

For example, if $J = \emptyset$ then the corresponding vector space $L^\natural_0 := L^\natural_{X^\emptyset}$ is $\Gamma(X, h(L))$ (see 2.5.3). We get a canonical embedding of $\mathcal{D}_{X^I}$-modules

$$L^\natural_0 \otimes \mathcal{O}_{X^I} \hookrightarrow L^\natural_{X^I}. \tag{3.7.6.4}$$

(ii) Denote by $L^\ell_{X^I}$ the cokernel of (3.7.2.4). Since X is affine one has $L^\ell_{X^I} = H^0 p_{I.} F_{\tilde{X^I}}$ where $F_{\tilde{X^I}} := (j_{I.} j_I^* DR(L) \boxtimes \mathcal{O}_{X^I})/(DR(L) \boxtimes \mathcal{O}_{X^I})$. The corresponding right $\mathcal{D}_{X^I}$-module equals to $p_{I*} \Phi_{\tilde{X^I}}$ where $\Phi_{\tilde{X^I}} := j_{I*} j_I^* (L \boxtimes \omega_{X^I})/L \boxtimes \omega_{X^I}$. In particular, $L^\ell_X = L^\ell$.

Notice that $L^\ell_{X^I}$, unlike $L^\natural_{X^I}$, depends on L in a purely X-local way, i.e., $L^\ell_{(x_i)}$ depends only on restriction of L to a neighbourhood of $\{x_i\}$.

For $S \in Q(I)$ consider the open subset $j^{[I/S]} : U^{[I/S]} \hookrightarrow X^I$ (see 3.5.4); for $s \in S$ let $p_s : X^I \to X^{I_s}$ be the projection, and $p_{\tilde{s}} := id_X \times p_s$. Decomposing $F_{\tilde{X^I}}$ by connected components of its support we get $j^{[I/S]*} \bigoplus_S p_{\tilde{s}}^* F_{\tilde{X^{I_s}}} \xrightarrow{\sim} j^{[I/S]*} F_{\tilde{X^I}}$ hence the isomorphism

$$j^{[I/S]*} \bigoplus_S p_s^* L^\ell_{X^{I_s}} \xrightarrow{\sim} j^{[I/S]*} L^\ell_{X^I}. \tag{3.7.6.5}$$

In particular, one has $j^{(I)*} \oplus p_i^* L^\ell \xrightarrow{\sim} j^{(I)*} L^\ell_{X^I}$.

Here is another interpretation of (3.7.2.5). For $s \in S$ consider the projection $q_s : X^I \to X^{I \smallsetminus I_s}$ and the coresponding embedding (3.7.6.3) of $\mathcal{D}$-modules $q_s^* L^\natural_{X^{I \smallsetminus I_s}} \hookrightarrow L^\natural_{X^I}$. We have a commutative diagram of embeddings of $\mathcal{D}_{X^I}$-modules

$$\begin{array}{ccccc} \prod_S p_s^* L^\natural_{X^{I_s}} & \longrightarrow & (L^\natural_{X^I})^S & \longleftarrow & L^\natural_{X^I} \\ \uparrow & & \uparrow & & \uparrow \\ (L^\natural_0 \otimes \mathcal{O}_{X^I})^S & \longrightarrow & \prod_S q_s^* L^\natural_{X^{I \smallsetminus I_s}} & \longleftarrow & L^\natural_0 \otimes \mathcal{O}_{X^I} \end{array} \tag{3.7.6.6}$$

The morphisms between the cokernels of vertical embeddings $\prod_S (p_s^* L^\natural_{X^{I_s}}/L_0^\natural \otimes \mathcal{O}_{X^{I_s}}) \to \prod_S L^\natural_{X^I}/L^\natural_{X^{I\smallsetminus I_s}} \leftarrow L^\natural_{X^I}/L_0^\natural \otimes \mathcal{O}_{X^I}$ being restricted to $U^{[I/S]}$ become isomorphisms; the composition is (3.7.6.5).

(iii) Assume that L is a Lie* algebra. Then $L^\natural_{X^I}$ is a Lie algebra in the tensor category $\mathcal{M}^\ell(X^I)$. One defines the Lie bracket on $L^\natural_{X^I}$ in the same way we defined the bracket on $h(L)$. Namely, the $*$ bracket on L yields a morphism $DR(L) \boxtimes DR(L) \to DR(\Delta_* L)$ on $X \times X$. Take the exterior tensor product of this morphism with $\mathcal{O}_{X^I}$, restrict it to $V \underset{X^I}{\times} V \subset (X \times X) \times X^I$, and push forward to X^I. We get a morphism of complexes of of left $\mathcal{D}_{X^I}$-modules $((p_I j_I).j_I^*(DR(L) \boxtimes \mathcal{O}_{X^I}))^{\otimes 2} \to (p_I j_I).j_I^*(\Delta^\cdot DR(\Delta_* L) \boxtimes \mathcal{O}_{X^I})$. The second term is canonically quasiisomorphic to $(p_I j_I).j_I^*(DR(L) \boxtimes \mathcal{O}_{X^I})$ (see 2.1.6(b)). Passing to H^0 we get a morphism $[\ \ ]^\natural : L^\natural_{X^I} \otimes L^\natural_{X^I} \to L^\natural_{X^I}$ which is the desired Lie bracket.

All the canonical morphisms in (i), (ii) above are compatible with our Lie algebra structure. In particular, $L^\natural$ is a Lie algebra in the tensor category of $\mathcal{D}$-modules on $\mathcal{R}(X)$.

3.7.7. Now we are ready to define our factorization algebra. Set

$$V_{X^I} := U(L^\natural_{X^I})/U(L^\natural_{X^I})L_0^\natural. \tag{3.7.7.1}$$

Here $U(L^\natural_{X^I})$ is the enveloping associative algebra of our Lie algebra $L^\natural_{X^I}$ in the abelian tensor k-category $\mathcal{M}^\ell(X^I)$. So V_{X^I} is *vacuum representation* - the $L^\natural_{X^I}$-module induced from the trivial $L_0^\natural \otimes \mathcal{O}_{X^I}$-module.

The construction of enveloping algebra and induced module is compatible with pull-backs. Therefore $U(L^\natural_{X^I})$ and V_{X^I} form left $\mathcal{D}$-modules $U(L^\natural)$ and V on $\mathcal{R}(X)$.

Our V has a canonical factorization structure. To define factorization isomorphisms κ (see 3.5.4.1) look at (3.7.6.6) and consider the corresponding morphisms of induced modules

$$\boxtimes_S V_{X^{I_s}} \longrightarrow \otimes_S U(L^\natural_{X^I})/U(L^\natural_{X^I}) q_s^* L^\natural_{X^{I\smallsetminus I_s}} \longleftarrow V_{X^I}. \tag{3.7.7.2}$$

The following well-known lemma (together with 3.7.6(ii)) shows that over $U^{[I/S]}$ both arrows in (3.7.7.2) are isomorphisms. This provides the factorization isomorphisms. The compatibilities from 3.5.4 obviously hold.

3.7.8 Lemma. Let $\phi : L_1 \to L_2$ be a morphism of Lie algebras over a commutative ring R, $N_1 \subset L_1$, $N_2 \subset L_2$ be Lie subalgebras such that $\phi(N_1) \subset N_2$ and $L_1/N_1 \xrightarrow{\sim} L_2/N_2$. Then the morphism $U(L_1)/U(L_1)N_1 \to U(L_2)/U(L_2)N_2$ is an isomorphism.

We give two proofs of the lemma. The first one is elementary. The second proof is more "immediate" but it uses a machinery too heavy for the purpose.[161]

First Proof. Set $P_i := U(L_i)/N_i$; let $v_i \in P_i$ be the "vacuum vectors"; $\pi : P_1 \to P_2$ our morphism.

(i) π is surjective. Indeed, π is compatible with the standard filtrations on P_i's. Since $\mathrm{gr}.P_i$ is a quotient of $\mathrm{Sym}^{\cdot}(L_i/N_i)$, $\mathrm{gr}\pi$ is surjective.

(ii) So to prove our lemma it suffices to construct a left inverse $P_2 \to P_1$ of π. To do this we will define an L_2-module structure on P_1 such that:

(a) The obvious L_1-action on P_1 equals the one coming from the L_2-action via π.

(b) The action of $N_2 \subset L_2$ kills v_1.

As follows from (b) there is a unique morphism of L_2-modules $P_2 \to P_1$ which sends v_2 to v_1. By (a) it is left inverse to π.

(iii) It remains to construct the L_2-action. We use the following remark: for any L_1-modules Q, R one has $\mathrm{Hom}_{L_1}(Q \otimes P_1, R) = \mathrm{Hom}_{L_1}(P_1, \mathrm{Hom}(Q, R)) = \mathrm{Hom}_{N_1}(Q, R)$.

Taking $Q = L_2, R = P_1$ we see that there is a unique morphism of L_1-modules $a : L_2 \otimes P_1 \to P_1$ with property $a(l_2 \otimes v_1) = l_1 v_1$ for every $l_i \in L_i$ such that $l_1 \mathrm{mod} N_1 = l_2 \mathrm{mod} N_2$.

Let us show that a is a Lie algebra action. We need to check coincidence of the two maps $\Lambda^2 L_2 \otimes P_1 \to P_1$. Both maps are L_1-equivariant, so (take $Q = \Lambda^2 L_2, R = P_1$ in the above remark) it suffices to show that their restrictions to $\Lambda^2 L_2 \otimes v_1$ are equal. This is clear, since both maps vanish on $\Lambda^2 N_2 \otimes v_1$, and on the image of $L_1 \otimes L_2 \otimes v_1$ they coincide by the L_1-equivariance.

Property (a) follows from the case $Q = L_1, R = P_1$ in the above remark; property (b) holds by construction. We are done.

Second Proof. Consider the derived version of the induced module construction. Namely,

[161] It will be of use in the proof of the Poincaré-Birkhoff-Witt theorem 3.7.14 though.

for an emedding of Lie algebras $N \subset L$ one may find Lie DG algebras $\tilde{L}, \tilde{N}$ placed in degrees ≤ 0 whose components are flat (e.g, free) R-modules, together with projections $\tilde{L} \to L$, $\tilde{N} \to N$ which are quasiisomorphisms, and morphisms of Lie DG algebras $\tilde{N} \to \tilde{L}$ that lifts the embedding $N \hookrightarrow L$. Consider the enveloping associative DG algebra $U(\tilde{L})$ as an $(\tilde{L}, \tilde{N})$-bimodule (with respect to the left and right actions). Denote by C the homological Chevalley complex of $\tilde{N}$ with coefficients in $U(\tilde{L})$ (where $\tilde{N}$ acts on $U(\tilde{L})$ by right multiplication). This is a DG $\tilde{L}$-module ($\tilde{L}$ acts by left multiplication) which sits in degrees ≤ 0. There is an obvious isomorphism $H^0 C \xrightarrow{\sim} U(L)/U(L)N$.

As a graded module C equals $U(\tilde{L}) \otimes \mathrm{Sym}(\tilde{N}[1])$. It carries a filtration $C_0 \subset C_1 \subset ..$ equal to the tensor product of the standard filtration $U(\tilde{L})_\cdot$ and the filtration $\mathrm{Sym}^{\leq \cdot}(\tilde{N}[1])$. This filtration is stable with respect to the differential, and there is a canonical isomorphism $\mathrm{gr}_i C = \mathrm{Sym}^i(\mathcal{C}one(\tilde{N} \to \tilde{L}))$. Therefore, as an object of the derived category of R-modules, $\mathrm{gr}_i C$ equals $\mathbb{L}\mathrm{Sym}^i(L/N)$.

Let us return to the situation of 3.7.8. Choose resolutions as above for both (L_1, N_1), (L_2, N_2) and morphisms of Lie DG algebras $\phi_{\tilde{L}} : \tilde{L_1} \to \tilde{L_2}$, $\phi_{\tilde{N}} : \tilde{N_1} \to \tilde{N_2}$ that lift ϕ and the diagram

$$\begin{array}{ccc} \tilde{L_1} & \xrightarrow{\phi_{\tilde{L}}} & \tilde{L_2} \\ \uparrow & & \uparrow \\ \tilde{N_1} & \xrightarrow{\phi_{\tilde{N}}} & \tilde{N_2} \end{array}$$

commutes. We get a morphism of filtered complexes $\phi_C : C_1 \to C_2$. Notice that $\mathrm{gr}_i(\phi_C)$ is the ith symmetric power of $\tilde{\phi} : \mathcal{C}one(\tilde{N_1} \to \tilde{L_1}) \to \mathcal{C}one(\tilde{N_2} \to \tilde{L_2})$. The condition of 3.7.4 assures that this $\tilde{\phi}$, hence $\mathrm{Sym}^\cdot \tilde{\phi}$, is a quasiisomorphism. Therefore ϕ_C is a quasiisomorphism. Passing to H^0 we get our lemma. □

Remark. The above lemma also shows that V_X has étale local nature. Namely, let $\pi : X' \to X$ be an étale map, $L_{X'}$ the π-pull-back of $L = L_X$, $V_{X'}$ the corresponding vacuum module. One has an obvious morphism of Lie algebras $\pi^* L_X^\natural \to L_{X'}^\natural$. It yields the morphism of vacuum modules $\pi^* V_X \to V_{X'}$ which is an isomorphism.

3.7.9 Lemma. Our V is a factorization algebra; its $\mathcal{D}$-module structure coincides with

the canonical one.

Proof. One needs only to check the flatness along codimension 1 diagonals. Indeed, the unit in V is the canonical generator 1 of the induced module V_X; since 1 is horizontal our $\mathcal{D}$-module structure is the canonical one.

Assume that $I = \{1, .., n\}$ and the codimension 1 diagonal $X^{n-1} \hookrightarrow X^n$ we consider is given by equation $x_{n-1} = x_n$. Let $j' : U' \hookrightarrow X^n$ be the complement to the union of diagonals $x_1 = x_n, .., x_{n-1} = x_n$, and projections $q : X^n \to X^{n-1}$, $p : X^n \to X$ are, respectively, $(x_1, .., x_n) \mapsto (x_1, .., x_{n-1})$, $(x_1, .., x_n) \mapsto x_n$.

The embeddings $q^* L^\natural_{X^{n-1}} \hookrightarrow L^\natural_{X^n} \hookleftarrow p^* L^\natural_X$ (see 3.7.6(i)) yield a short exact sequence

$$0 \to q^* L^\natural_{X^{n-1}} \to L^\natural_{X^n} \to j'_* j'^* p^* L^\ell_X \to 0. \tag{3.7.9.1}$$

Consider the canonical filtration on the enveloping algebra $U(L^\natural_{X^n})$. The Poincaré-Birkhoff-Witt theorem $\mathrm{gr}U = \mathrm{Sym}$ holds for Lie algebras in any abelian tensor $\mathbb{Q}$-category (see, e.g., [DM] 1.3.7), so $\mathrm{gr}_a U(L^\natural_{X^n}) = \mathrm{Sym}^a L^\natural_{X^n}$. Let us consider (3.7.9.1) as a two step increasing filtration on $L^\natural_{X^n}$. Notice that the sub- and quotient modules in our filtration are Tor-independent. Therefore it defines on every $\mathrm{Sym}^a L^\natural_{X^n}$ an increasing filtration with successive quotients $q^*(\mathrm{Sym}^{a-m} L^\natural_{X^{n-1}}) \otimes j'_* j'^* p^*(\mathrm{Sym}^m L^\ell_X)$. Set $Q := U(L^\natural_{X^n})/q^* U(L^\natural_{X^{n-1}})$; we see that $Q = j'_* j'^* Q$.

The Lie algebra $L^\natural_0 \subset q^* L^\natural_{X^{n-1}} \subset L^\natural_{X^n}$ acts on $U(L^\natural_{X^n})$ by right multiplications, and V_{X^n} is the $\mathcal{D}_{X^n}$-module of coinvariants $U(L^\natural_{X^n})_{L^\natural_0}$. The short exact sequence of $L^\natural_0$-modules $0 \to q^* U(L^\natural_{X^{n-1}}) \to U(L^\natural_{X^n}) \to Q \to 0$ yields the short exact sequence of coinvariants

$$0 \to q^* V_{X^{n-1}} \to V_{X^n} \to Q_{L^\natural_0} \to 0. \tag{3.7.9.2}$$

Indeed, one needs only check the exactness from the left which follows from the fact that every morphism from a $j'_* \mathcal{O}_{U'}$-module to $q^* V_{X^{n-1}}$ vanishes.

Both sub- and quotient modules in (3.7.9.2) are flat along our diagonal $X^{n-1} \subset X^n$ (recall that $Q_{L^\natural_0} = j'_* j'^* Q_{L^\natural_0}$), so same is true for the middle term. We are done. □

3.7.10. Consider the chiral algebra V^r that corresponds to our factorization algebra V. So V^r is the right $\mathcal{D}$-module corresponding to the left $\mathcal{D}_X$-module $V_X = U(L^\natural_X)/U(L^\natural_X) L^\natural_0$.

The chiral product $\mu : j_*j^*V_X \boxtimes V_X \to \Delta_*V_X$ may be described explicitly as follows. The exact sequence $0 \to p_2^*L_X^\natural \to L_{X\times X}^\natural \to j_*j^*p_1^*L_X^\ell \to 0$ (see (3.7.9.1)) shows that the embedding $p_1^*L_X^\natural \hookrightarrow L_{X\times X}^\natural$ yields an isomorphism between the induced modules $j^*p_1^*V_X \underset{\sim}{\longrightarrow} j^*U(L_{X\times X}^\natural)/U(L_{X\times X}^\natural)p_1^*L_X^\natural$. In particular, $L_{X\times X}^\natural$ acts on $j_*j^*p_1^*V_X$; same for $j_*j^*p_2^*V_X$. Therefore the tensor product $j_*j^*V_X \boxtimes V_X$ is an $L_{X\times X}^\natural$-module. There is a canonical isomorphism of $L_{X\times X}^\natural$-modules $j_*j^*V_{X\times X} \underset{\sim}{\longrightarrow} j_*j^*V_X \boxtimes V_X$ which sends the generator $1_{X\times X}$ to $1_X \boxtimes 1_X$. Our chiral product μ is composition of the inverse to this isomorphism and projection $j_*j^*V_{X\times X} \to (j_*j^*V_{X\times X})/V_{X\times X} = \Delta_*V_X$ where the letter identification is the structure isomorphism $\Delta^*V_{X\times X} = V_X$. Notice that μ is a morphism of $L_{X\times X}^\natural$-modules.

The morphism $L_X^\natural \to V_X, l \mapsto l\cdot 1$, kills $L_0^\natural$, so it yields a morphism of right $\mathcal{D}_X$-modules $i : L \to V^r$. The next proposition finishes the proof of 3.7.1:

3.7.11 Proposition. The above $i : L \to V^r$ is a morphism of Lie* algebras. It satisfies the universality property: for every chiral algebra A the map $\mathrm{Hom}_{\mathcal{CA}(X)}(V^r, A) \to \mathrm{Hom}_{\mathcal{L}ie^*(X)}(L, A^{(*)})$, $\varphi \mapsto \varphi i$, is bijective.

Proof. We begin with a useful general construction.

Let A be a chiral algebra, M an A-module. Consider the Lie algebra[162] $A_X^\natural := A_X^{(*)\natural}$ in the tensor category $\mathcal{M}^\ell(X)$ (see 3.7.6). One has $H^0p_{2*}j_*j^*A \boxtimes M = H^0(p_{2*}j_*j^*A \boxtimes \omega_X) \otimes^! M = A_X^\natural \otimes M$; here $p_2 : X \times X \to X$ is the second projection. Therefore we get a canonical morphism

$$\bullet = \bullet_M := p_{2*}(\mu_M) : A_X^\natural \otimes M \to M. \tag{3.7.11.1}$$

Exercise: check that this is a Lie algebra action of $A_X^\natural$ on the $\mathcal{D}$-module M. On the Lie subalgebra $A_0^\natural = H^0_{DR}(X, A) = \Gamma(X, h(A)) \subset A_X^\natural$ this action coincides with the usual $h(A)$-action for the $A^{(*)}$-module structure on M (see 2.5.4).

Notice that $\bullet_M$ determines μ_M uniquely (since p_{2*} is right exact and for $\mathcal{D}_{X\times X}$-modules supported on the diagonal it is fully faithful).

[162] Specialists in vertex operator algebras sometimes call a similar object "the Lie algebra of local fields".

In particular, we have a canonical action of $A_X^\natural$ on A. The morphism $A_X^\natural \to A^\ell, a^\natural \mapsto a^\natural \bullet 1$, coincides with the standard projection $A_X^\natural \to A_X^\natural/A_0^\natural \otimes \mathcal{O}_X = A^\ell$ (see 3.7.6(ii)). A morphism of chiral algebras $A \to A'$ is the same as a morphism of $\mathcal{D}$-modules $\varphi : A \to A'$ which is compatible with $\bullet$ actions (i.e., $\varphi \bullet_A = \bullet_{A'}(\varphi^\natural \otimes \varphi)$) and sends 1_A to $1_{A'}$.

Let us return to our chiral algebra V^r. Let $\bullet_L : L_X^\natural \otimes V_X \to V_X$ be the morphism $\bullet_{V^r}$ composed with $i^\natural : L_X^\natural \to V_X^{r\natural}$.

Sublemma. $\bullet_L$ coincides with our old canonical $L^\natural$-action on V_X.

Proof of Sublemma. We know that the chiral product $\mu : j_*j^*V_X \boxtimes V_X \to \Delta_*V_X$ is a morphism of $L_{X\times X}^\natural$-modules (see 3.7.10). Restricting this action to $p_2^*L_X^\natural \subset L_{X\times X}^\natural$ and pushing μ forward by p_2 we see that $\bullet : V_X^{r\natural} \otimes V_X \to V_X$ is a morphism of $L_X^\natural$-modules. Here $L_X^\natural$ acts on $V_X^{r\natural} := H^0 p_{2*}j_*j^*p_1^*V_X$ via the embedding $p_2^*L_X^\natural \hookrightarrow L_{X\times X}^\natural$ and the $L_{X\times X}^\natural$-action on $j_*j^*p_1^*V_X = U(L_{X\times X}^\natural)/U(L_{X\times X}^\natural)p_2^*L_X^\natural$ (see 3.7.10), and on V_X in our old canonical way (as on the vacuum module). One checks easily that $i^\natural : L_X^\natural \to V_X^{r\natural}$ is a morphism of $L_X^\natural$-modules. So for $l, l' \in L_X^\natural$, $v \in V_X$ one has $l'(l \bullet_L v) = [l', l] \bullet_L v + l \bullet_L (l'v)$. Since $l \bullet_L 1 = l1$ and 1 generates V_X as an $L_X^\natural$-module, we are done. □

End of proof of 3.7.11. Take $l \in L_0^\natural \subset L_X^\natural$, $l' \in L$. By sublemma, $l \bullet_L i(l') = i(ad_l l')$. This shows that i is a morphism of Lie* algebras (see 2.5.2). Our sublemma also implies that V^r is generated by $i(L)$ as a chiral algebra, so the map $\mathrm{Hom}_{\mathcal{CA}(X)}(V^r, A) \to \mathrm{Hom}_{\mathcal{L}ie^*(X)}(L, A^{(*)})$ is injective. Let us define its inverse. Let $\psi : L \to A^{(*)}$ be a morphism of Lie* algebras. It yields a morphism of Lie algebras $\psi^\natural : L_X^\natural \to A_X^\natural$, hence $\bullet_A$ defines an $L_X^\natural$-action on A. Since $L_0^\natural \subset L_X^\natural$ kills 1_A we get a morphism of $L_X^\natural$-modules $\tilde{\psi} : V_X \to A^\ell$ which sends 1 to 1_A. One has $\tilde{\psi} i = \psi$, so it remains to check that $\tilde{\psi}$ is a morphism of chiral algebras. Since it is a morphism of $L_X^\natural$-modules, we know that $\mu_A(\tilde{\psi}^{\boxtimes 2})$ equals $\tilde{\psi}\mu_{V^r}$ on the image of $j_*j^*L \boxtimes V^r \to j_*j^*V^r \boxtimes V^r$. Since the image of L generates V^r as a chiral algebra, we are done. □

3.7.12. Let L_α be a family of Lie* algebras. Consider the Lie* algebra $\oplus L_\alpha$. It follows immediately from the universality property of $\otimes$ and U (see 3.5.16, 3.7.1) that one has a

canonical isomorphism of chiral algebras

$$U(\oplus L_\alpha) \xrightarrow{\sim} \otimes U(L_\alpha). \tag{3.7.12.1}$$

In particular, for any Lie* algebra L the diagonal map $L \to L \times L$ yields a coproduct morphism $U(L) \to U(L) \otimes U(L)$ which makes $U(L)$ a chiral Hopf algebra (see 3.5.18).

3.7.13. We are going to prove a chiral version of the Poincaré-Birkhoff-Witt theorem.

Let A be a chiral algebra equipped with a filtration $A_0 \subset A_1 \subset .. \subset A$ (see 3.3.11). Notice that A_0 is a chiral subalgebra of A, and A_1 is an A_0-submodule. We say that our filtration $A.$ is *1-generated* if all the arrows $j_*j^*A_i \boxtimes A_j \to \Delta_* A_{i+j}$, $i, j \geq 0$, are surjective. If $R \subset A$ is a chiral subalgebra of A and $M \supset R$ an R-submodule of A that generates A then there is a unique 1-generated filtration $A.$ such that $A_0 = R$, $A_1 = M$. Namely, $A_{i+1} := \mu(j_*j^*A_1 \boxtimes A_i)$ for $i \geq 0$. We say that this filtration is generated by (R, M); if $R = \omega_X \cdot 1$, then we simply say that our filtration is generated by M. Assume that M is a Lie* subalgebra of A,[163] R is commutative, and M normalizes R. Then our filtration is commutative,[164] and the obvious surjective morphism of $\mathbb{Z}$-graded commutative $\mathcal{D}_X$-algebras $\mathrm{Sym}^{\cdot} M^\ell \to \mathrm{gr}.A^\ell$ is compatible with the coisson brackets.

Let us return to our situation. Denote by L^{tor} the $\mathcal{O}_X$-torsion submodule of L. Since any binary chiral operation on an $\mathcal{O}$-torsion $\mathcal{D}$-module vanishes the canonical morphism $L \to U(L)$ kills the commutator $[L^{tors}, L^{tors}]$ (which is a Lie ideal in L). Therefore $U(L) \xrightarrow{\sim} U(L/[L^{tor}, L^{tor}])$.

The chiral algebra $U(L)$ is generated by the image of L which is a Lie* subalgebra of $U(L)$. The corresponding filtration $U(L).$ is called *the standard*, or *Poincaré-Birkhoff-Witt, filtration*. We have a canonical surjective morphism of coisson algebras

$$\mathrm{Sym}^{\cdot} L^\ell \twoheadrightarrow \mathrm{gr}.U(L)^\ell \tag{3.7.13.1}$$

which factors through

$$\mathrm{Sym}^{\cdot}(L/[L^{tor}, L^{tor}])^\ell \twoheadrightarrow \mathrm{gr}.U(L)^\ell. \tag{3.7.13.2}$$

[163] The next conditions hold automatically if $R = \omega_X \cdot 1$.

[164] See 3.3.11.

3.7.14 Theorem. (3.7.13.2) is an isomorphism. In particular, if L is $\mathcal{O}_X$-flat then (3.7.13.1) is an isomorphism.

Proof. We can assume that X is affine. Consider the projection $U(L^\natural_X) \to V_X = U(L)^\ell$. As follows from the proof of 3.7.11 (see the Sublemma) our standard filtration equals to the image of the usual standard filtration on the enveloping algebra $U(L^\natural_X)$.

Consider the filtered $\mathcal{O}_X$-complex C from the second proof of 3.7.8 for the pair $L^\natural_0 \subset L^\natural_X$. Then H^0C equals $U(L)^\ell$ (as a filtered module) and $\mathrm{gr}_i C = \mathbb{L}\mathrm{Sym}^i_{\mathcal{O}_X}(L^\ell)$. Therefore one has $H^0\mathrm{gr}_i C = \mathrm{Sym}^i L^\ell$, $H^{-1}\mathrm{gr}_i C = \mathrm{Sym}^{i-2}(L/L^{tor})^\ell \otimes \mathrm{Tor}_1^{\mathcal{O}_X}(L^{tor\ell}, L^{tor\ell})^{\Sigma_2} = \mathrm{Sym}^{i-2}(L/L^{tor})^\ell \otimes \Delta^!(L^{tor} \boxtimes L^{tor})^{\ell\, sgn}$. The differential $d_1^{-2,1} : H^{-1}\mathrm{gr}_2 C \to H^0\mathrm{gr}_1 C$ in the spectral sequence coincides with the Lie* bracket $\Delta^!(L^{tor} \boxtimes L^{tor})^{\ell\, sgn} \to L^{tor\ell} \subset L^\ell$.[165] Now recall that C is a module over some Lie algebra whose quotient is $L^\natural_X$. This implies that the image of the differential $d_1 : H^{-1}\mathrm{gr}C \to H^0\mathrm{gr}C$ is the ideal generated by the image of $d_1^{-2,1}$, and all higher $d_r^{p,q}$'s for $p+q=-1$ vanish (since they all vanish on $\mathrm{Ker} d_1^{-2,1}$). We are done. □

We are going to describe $U(L)$-modules in more explicit terms (contrary to the usual situation $U(L)$-modules are *not* the same as L-modules).

3.7.15 Proposition. The obvious functor $\mathcal{M}(X, U(L)) \to \mathcal{M}(X, L)$ admits a left adjoint induction functor

$$\mathrm{Ind} = \mathrm{Ind}_L : \mathcal{M}(X, L) \to \mathcal{M}(X, U(L)).$$

Proof. For a $\mathcal{D}$-module M an L-action on it amounts to a Lie* algebra structure on $L \oplus M$ such that the bracket on M vanishes and $L \hookrightarrow L \oplus M$ is a morphism of Lie* algebras. So an L-module M yields a chiral algebra $U(L \oplus M)$. The $\mathbb{G}_m$-action on M by homotheties defines a $\mathbb{Z}$-grading on this chiral algebra. Its zero component is $U(L)$. The degree 1 component is $\mathrm{Ind}M$. To check the universality property use 3.3.5. □

Remark. The functor Ind is compatible with étale localization of X.

[165] To check this we can assume that $L = L^{tor}$ and it is supported at a single point $x \in X$, i.e., $L = i_{x*}F$ for some Lie algebra F. Then $L^\natural = j_{x*}(F \otimes \mathcal{O}_{U_x})$, $L^\natural_0 = F \otimes \mathcal{O}_X$ and C is the Chevalley complex of F with coefficients in $U(F) \otimes j_{x*}\mathcal{O}_{U_x}$. Now the statement is clear.

Let us describe the induced module $\mathrm{Ind}M$ explicitly. We can assume that X is affine. Consider the Lie algebra $L^\natural_X$ defined in 3.7.6(iii). The Lie* action of L on M yields an action on M of the Lie subalgebra $L^\natural_0 \subset L^\natural_X$. Now one has a canonical isomorphism

$$\mathrm{Ind}M \xrightarrow{\sim} U(L^\natural_X) \underset{U(L^\natural_0)}{\otimes} M. \tag{3.7.15.1}$$

3.7.16. Define a *chiral L-module* as a $\mathcal{D}_X$-module M equiped with a chiral pairing $\mu = \mu_{(L,M)} \in P_2^{ch}(\{L,M\},M)$ (called a *chiral action* of L on M) which satisfies the following "truncated" version of the Lie algebra action axiom. Let $j' : U' \hookrightarrow X \times X \times X$ be the complement to the diagonals $x_1 = x_3$ and $x_2 = x_3$. Consider a chiral operation $\mu_{(L_1,M)}(id_{L_1}, \mu_{(L_2,M)}) - \mu_{(L_2,M)}(id_{L_2}, \mu_{(L_1,M)}) \in P_3^{ch}(\{L,L,M\},M)$ and its restriction $\nu : j'_* j'^*(L \boxtimes L \boxtimes M) \to \Delta^{(3)}M$. We demand that ν equals to composition

$$j'_* j'^*(L \boxtimes L \boxtimes M) \to \Delta_*^{1=2} j_* j^*(L \boxtimes M) \to \Delta_*^{(3)} M. \tag{3.7.16.1}$$

Here $\Delta^{1=2} : X \times X \hookrightarrow X \times X \times X$ is the embedding $(x,y) \mapsto (x,x,y)$, the first arrow is $[\quad]_L \boxtimes id_M$, the second one is $\Delta_*^{1=2}(\mu_{(L,M)})$.

Example. Assume that M is supported at a point $x \in X$. Then a chiral L-action on M is the same as a $*$ action of the Lie* algebra $j_{x*}j_x^*L$ on M. Here $j_x : U_x \hookrightarrow X$ is the complement to x.

Assume that X is affine. As in the beginning of the proof of 3.7.11, a chiral pairing $\mu_{(L,M)}$ amounts to a morphism $\bullet_\mu : L^\natural_X \otimes M \to M$ such that its restriction to $L^\natural_0 \otimes M = \Gamma(X, h(L)) \otimes M$ comes from a $*$ operation $\in P_2^*(\{L,M\},M)$.

3.7.17 Lemma. $\mu_{(L,M)}$ is a chiral action if and only if $\bullet_\mu$ is an action of the Lie algebra $L^\natural_X$ on M.

Proof. Left to the reader. □

Denote by $\mathcal{M}(X, L^{ch})$ the category of chiral L-modules on X. A chiral $U(L)$-module structure defines in the obvious manner a chiral L-action on M. So we have a functor

$$\mathcal{M}(X, U(L)) \to \mathcal{M}(X, L^{ch}). \tag{3.7.17.1}$$

3.7.18 Proposition. Functor (3.7.17.1) is an equivalence of categories.

Proof. Left to the reader. □

Let $x \in X$ be a point, $j_x : U_x \hookrightarrow X$ its complement. The topological associative algebra $U(L)^{as}_x$ (see 3.4.2–3.4.4) can be described as follows. This algebra depends only on restriction of $U(L)$ to U_x (see Remark in 3.4.4), so it is convenient to change notation for the moment and assume that L is a Lie* algebra on U_x. Consider the topological Lie algebra $L_{(x)} := \hat{h}^{Lie}_x j_{x*}L$ (see 2.5.12). Denote by $U(L_{(x)})$ the topological enveloping algebra of $L_{(x)}$ defined as completion of the plain enveloping algebra with respect to topology whose base of opens is formed by left ideals generated by an open subspace in $L_{(x)}$.

3.7.19 Proposition. There is a canonical isomorphism of topological associative algebras

$$U(L_{(x)}) \xrightarrow{\sim} U(L)^{as}_x.$$

Proof. $L_{(x)}$ and $U(L)^{as}_x$ are completions of, respectively, $h_x j_{x*}L$ and $h_x j_{x*}U(L)$ with respect to certain topologies. It follows from definitions that the canonical morphism $h_x j_{x*}L \to h_x j_{x*}U(L)$ is continuous with respect to these topologies, thus we have a continuous morphism $L_{(x)} \to U(L)^{as}_x$. This is a morphism of Lie algebras (as follows from Exercise in 3.4.4). So it yields a morphism of topological associative algebras $U(L_{(x)}) \to U(L)^{as}_x$. According to 2.5.15, 3.7.18, 3.4.5, and Example in 3.7.16 this morphism yields an equivalence between the categories of discrete modules. Since our algebras are complete and separated, and their topologies are defined by systems of left ideals, this shows that our morphism of topological algebras is an isomorphism. □

Let us describe some variants of the enveloping algebra construction.

3.7.20. Let $L^\flat$ be an (automatically central) extension of a Lie* algebra L by ω_X. Let $1^\flat$ be the corresponding horizontal section of $L^{\flat\ell}$ (so $L^\ell = L^{\flat\ell}/\mathcal{O}_X 1^\flat$). The $\flat$-*twisted* chiral enveloping algebra of L is the quotient $U(L)^\flat$ of $U(L^\flat)$ modulo the ideal generated by $1 - 1^\flat$. So for any chiral algebra A a morphism $U(L)^\flat \to A$ amounts to a morphism of Lie* algebras $L^\flat \to A^{(*)}$ which maps $1^\flat$ to 1_A.

As in 3.7.13, the image of $L^\flat$ in $U(L)^\flat$ defines the standard filtration on $U(L)^\flat$, and one has a canonical surjective map of coisson algebras

$$\mathrm{Sym}^{\cdot} L^\ell \twoheadrightarrow \mathrm{gr}.U(L)^\flat. \tag{3.7.20.1}$$

One has the twisted version of PBW:

3.7.21 Proposition. If L is $\mathcal{O}_X$-flat then (3.7.20.1) is an isomorphism.

Proof. Notice that $\varepsilon := 1 - 1^\flat$ is a central horizontal section of $U(L^\flat)^\ell$. It defines the endomorphism $\varepsilon\cdot$ of $\mathcal{D}$-module $U(L^\flat)$, and $U(L)^\flat$ is its cokernel. This endomorphism shifts the standard filtration of $U(L^\flat)$ by 1, and the corresponding endomorphism of $\mathrm{gr}U(L^\flat)^\ell = \mathrm{Sym}L^{\flat\ell}$ (see 3.7.14) is multiplication by $1^\flat \in L^\ell$. It is injective with cokernel equal to $\mathrm{Sym}L^\ell$. This implies our PBW. □

The morphism $L^\flat \to U(L)^\flat$ yields the obvious pseudo-tensor functor $\mathcal{M}(X, U(L)^\flat) \to \mathcal{M}(X, L)$ (since any $*$ action of ω_X on any module is trivial $L^\flat$-modules are the same as L-modules). It admits a left adjoint functor $\mathrm{Ind}^\flat = \mathrm{Ind}^\flat_L : \mathcal{M}(X, L) \to \mathcal{M}(X, U(L)^\flat)$ (see 3.7.15). For an L-module M we have $\mathrm{Ind}^\flat_L M = \mathrm{Ind}_{L^\flat} M/(1 - 1^\flat)\mathrm{Ind}_{L^\flat} M$. One defines the canonical filtration on $\mathrm{Ind}^\flat M$ as the image of the canonical filtration on $\mathrm{Ind}_{L^\flat} M$, so we have a canonical morphism of graded $\mathrm{Sym}L^\ell$-modules

$$\mathrm{Sym}^{\cdot} L^\ell \otimes M \twoheadrightarrow \mathrm{gr}.\mathrm{Ind}^\flat M. \tag{3.7.21.1}$$

3.7.22 Lemma. If L is $\mathcal{O}_X$-flat then (3.7.21.1) is an isomorphism.

Proof. Same as that of 3.7.21 (use ??). □

Here is the twisted version of compatibility 3.7.19. Assume that we are in the situation of 3.7.19, so L is a Lie* algebra on U_x and let $L^\flat$ be an extension of L by ω_{U_x}. Let us compute the topological associative algbera $U(L)^{\flat as}_x = (U(L)^\flat)^{as}_x$.

The morphism of topological Lie algebras $L^\flat_{(x)} \twoheadrightarrow L_{(x)}$ is continuous and open. Its kernel equals the image of $i_{(x)} : k = \hat{h}_x j_{x*}\omega_{U_x} \to L^\flat_{(x)}$. The twisted topological enveloping algebra $U(L_{(x)})^\flat$ is the quotient of $U(L^\flat_{(x)})$ modulo the closed ideal generated by the central element $1^\flat - 1$ where $1^\flat := i_{(x)}(1)$.

3.7.23 Proposition. There is a canonical isomorphism of topological associative algebras

$$U(L_{(x)})^\flat \xrightarrow{\sim} U(L)_x^{\flat as}.$$

Proof. The composition of the morphisms $U(L^\flat_{(x)}) \to U(L^\flat)^{as}_x \to U(L)^{\flat as}_x$ sends $1^\flat$ to 1, so it yields a morphism $U(L_{(x)})^\flat \to U(L)^{\flat as}_x$. As follows from 3.7.19 this morphism induces an equivalence between the categories of discrete modules, hence it is an isomorphism (use the fact that our algebras are complete and separated, and their topologies are defined by systems of left ideals). □

3.7.24. Let A be a chiral algebra and L a Lie* algebra that acts on A (see 3.3.3). For a chiral algebra B consider the set of pairs (i, ψ) where $i: A \to B$ is a morphism of chiral algebras and $\psi: L \to B$ a morphism of Lie* algebras, such that i is a morphism of L-modules (here L acts on B via ψ and the adjoint action of B). Such pairs depend on B functorially, and it is easy to see that there exists a universal pair (i, ψ); the corresponding chiral algebra B is denoted by $A \circledast U(L)$. Indeed, consider the L-action on A as an action of L on the Lie* algebra $A^{(*)}$,[166] so we have the semi-direct product of Lie* algebras L and $A^{(*)}$. Our $A \circledast U(L)$ is the quotient of its chiral enveloping algebra modulo the obvious relations.[167]

Here is another construction of $A \circledast U(L)$. Recall that Ind_L^{ch} is a pseudo-tensor functor, so $\mathrm{Ind}_L^{ch} A$ is a Lie^{ch} algebra. The image of 1_A is the unit in $\mathrm{Ind}_L^{ch} A$, so $\mathrm{Ind}_L^{ch} A$ is a chiral algebra. The canonical morphism $i: A \to \mathrm{Ind}_L^{ch} A$ and the morphism $\psi: L \to \mathrm{Ind}_L^{ch} A$ that comes from the chiral action of L on 1_A satisfy the above properties. One checks immediately the universality property, so $A \circledast U(L) = \mathrm{Ind}_L A$.

We leave it to the reader to check that the canonical filtration on $\mathrm{Ind}_L A$ is a chiral algebra filtration, and the canonical morphism

$$A \otimes \mathrm{Sym}^{\cdot} L^\ell \to \mathrm{gr.}(A \circledast U(L)) \tag{3.7.24.1}$$

is a morphism of chiral algebras. According to ?? if L is $\mathcal{O}_X$-flat then (3.7.24.1) is an isomorphism.

[166] We use notation of 3.3.2.

[167] Which say that the morphism $A \to A \circledast U(L)$ is a morphism of chiral algebras, not just of Lie* algebras.

3.7.25. The constructions of 3.7.24, 3.7.20 easily combine. Namely, assume that we have A, L as in 3.7.24 and $L^\flat$ as in 3.7.20. Look at pairs $i: A \to B$, $\psi^\flat: L^\flat \to B$ that satisfy same conditions as i, ψ from 3.7.24 plus we assume that $\psi^\flat(1^\flat) = 1_B$. As in 3.7.24 there exists a universal pair $(i, \psi^\flat)$, and we denote the corresponding B by $A \circledast U(L)^\flat$. One has $A \circledast U(L)^\flat = A \circledast U(L^\flat)/(1-1^\flat)A \circledast U(L^\flat)$. There is a canonical identification $A \circledast U(L)^\flat = \mathrm{Ind}_L^\flat A$. The canonical filtration on $\mathrm{Ind}_L^\flat A$ is a chiral algebra filtration, and the canonical morphism $A \otimes \mathrm{Sym}^\cdot L^\ell \to \mathrm{gr.}(A \circledast U(L)^\flat)$ is a morphism of chiral algebras. If L is $\mathcal{O}_X$-flat then this is an isomorphism (see 3.7.22). This version of the PBW theorem will be of use in Section 3.8.

3.7.26. In the rest of this section we give a different prove of the PBW theorem 3.7.14 which does not use the factorization algebra picture but followss the approach of 3.7.2–3.7.4. It will not be used in the rest of this work.

We get control on $U(L)$ by means of a canonical resolution $\mathcal{C}(L)$ defined in 3.7.27. Its definition uses Quillen's adjoint functors identifying the derived categories of DG Lie algebras and DG cocommutative coalgebras [QR]. Recall the definitions.

Let $\mathcal{A}$ be a tensor $\mathbb{Q}$=category (we do not assume it to be unital) which admits countable direct sums, so we have the tensor category $C\mathcal{A}$ of complexes in $\mathcal{A}$. Consider the categories $\mathcal{L}ie(C\mathcal{A})$, $\mathcal{C}oc(C\mathcal{A})$ of Lie DG algebras, resp. cocommutative and coassociative DG coalgebras in $\mathcal{A}$.

For $L \in \mathcal{L}ie(C\mathcal{A})$, $B \in \mathcal{C}oc(C\mathcal{A})$ a *Maurer-Cartan map* $\varphi : B \to L$ is a morphism of graded objects $B \to L[1]$ such that $d\varphi + \varphi d = \frac{1}{2}[\varphi, \varphi]$. Here $[\varphi, \varphi]$ is the composition $B \to B \otimes B \xrightarrow{\varphi \otimes \varphi} L \otimes L \to L$, the first arrow is the coproduct, the last one is the bracket. Denote by $\mathrm{MC}(B, L)$ the set of Maurer-Cartan maps; it depends on B, L in functorial way.

Both functors $\mathrm{MC}(B, \cdot)$, $\mathrm{MC}(\cdot, L)$ are representable. Denote the corresponding objects by $F(B) \in \mathcal{L}ie(C\mathcal{A})$, $G(L) \in \mathcal{C}oc(C\mathcal{A})$. They can be described explicitely as follows. Consider the universal Maurer-Cartan maps as morphisms of graded objects $B[-1] \to F(B)$, $G(L) \to L[1]$. They identify $F(B)$, $G(L)$ as plain graded objects with, respectively, the free Lie graded algebra generated by $B[-1]$ and the cofree cocommutative graded

algebra cogenerated by $L[1]$. The differentials are determined on generators by the Maurer-Cartan property.

The functor $F : \mathcal{C}oc(C\mathcal{A}) \to \mathcal{L}ie(C\mathcal{A})$ is left adjoint to $G : \mathcal{L}ie(C\mathcal{A}) \to \mathcal{C}oc(C\mathcal{A})$ in the obvious way.

Remark. $G(L)$ is the truncated Chevalley complex of L (if $L \in \mathcal{L}ie(\mathcal{A})$ then $G(L)^i = \Lambda^{-i}L$ if $i < 0$ and 0 otherwise).

3.7.27. Let us return to our situation, so we have a Lie* algebra L for which we want to check the PBW property. Consider L as a Lie* algebra on X^{S} (see 3.7.3). Denote by $\mathcal{C}(L)$ the Lie^{ch} algebra on X^{S} defined as the image of L by the composition of functors

$$\mathcal{L}ie^*(X^{\mathrm{S}}) \xrightarrow{G^*} \mathcal{C}oc^*(X^{\mathrm{S}}) \hookrightarrow \mathcal{C}oc^{ch}(X^{\mathrm{S}}) \xrightarrow{F^{ch}} \mathcal{L}ie^{ch}(X^{\mathrm{S}}). \tag{3.7.27.1}$$

Here the indices $*$, ch show that we deal with either of the two tensor category structures on $\mathcal{M}(X^{\mathrm{S}})$ (see 3.7.3); the middle functor is the obvious embedding (any $*$ coalgebra on X^{S} is automatically a ch coalgebra by (3.7.3.3)).

One has $\mathcal{C}(L)^{>0} = 0$ and $\mathcal{C}(L)^0$ is the free Lie algebra in $\mathcal{M}(X^{\mathrm{S}})^{ch}$ generated by $G^*(L)^{-1} = L$. An immediate calculation shows that the image of $\mathcal{C}(L)^{-1} \xrightarrow{d} \mathcal{C}(L)^0$ equals the ideal $\mathcal{I}$ from the end of 3.7.3, therefore one has (here $U^{\mathrm{S}} := U^{\mathrm{S}}_{\mathcal{L}ie}$)

$$H^0\mathcal{C}(L) = U^{\mathrm{S}}(L). \tag{3.7.27.2}$$

3.7.28. Let us compute $\mathcal{C}(L)$ explicitly. At the moment we forget about the differentials. One has $G^*(L) = \oplus (L[1])^{\overset{*}{\otimes}{}^n_{\Sigma_n}}$

§8 BRST, ALIAS SEMI-INFINITE, HOMOLOGY

Below "$\mathcal{D}$-module", etc., always means "super $\mathcal{D}$-module" (see 1.1.16 for "super" conventions). As in 1.4.20 *for a graded $\mathcal{D}$-module M we set $\hbar(M) := h(M^{0even})$.*

We discuss BRST homology in the setting of chiral algebras (3.8.1–3.8.16) and in the more traditional setting of Tate's linear algebra (3.8.17–3.8.22). The two pictures are compared in 3.8.23–3.8.25.

The BRST homology construction is a neat quantum version of the classical Hamiltonian reduction. It originated in mathematical physics (BRST is for physicists Becchi-Rouet-Stora-Tyupin), came to representation theory in a pioneering work of B.Feigin [F] (his term "semi-infinite" has to do with decomposition of a Tate vector space in a sum of compact and discrete "halves"), and was further developed in [FHZ]. Our exposition of the linear algebra picture follows these references except that we tried to get rid of many redundant structures (gradings, normal orderings, explicit cocycles, etc.)

3.8.1 Chiral Weyl algebras. Let T be a $\mathcal{D}_X$-module; we assume that it has no $\mathcal{O}$-torsion. Let $<>_T \in P_2^*(\{T,T\},\omega_X)$ be a skew-symmetric[168] $*$ pairing. Set $T^\flat := T \oplus \omega_X$. This is a Lie* algebra with commutator equal to $<,>_T$ (plus zero components). So $T^\flat$ is an extension of a commutative Lie* algebra T by ω_X. Let $\mathcal{W} = \mathcal{W}(T, <>_T)$ be the corresponding twisted enveloping chiral algebra $U(T)^\flat$ (see 3.7.20). This is the *Weyl algebra* of $(T, <>_T)$. It carries the standard filtration so that $T^\flat = \mathcal{W}_1$, and gr.$\mathcal{W} = \mathrm{Sym}^\cdot T$ (see 3.7.21).

The *classical*, or *coisson*, Weyl algebra $\mathcal{W}^c$ is Sym$T$ equipped with coisson bracket that equals to $<>_T$ on $T = \mathrm{Sym}^1 T$. It differs from gr.$\mathcal{W}$ as coisson algebra: the coisson bracket on gr.$\mathcal{W}$ is trivial.

3.8.2. Assume our T is a $\mathcal{D}_X$-vector bundle and $<>_T$ is non-degenerate, i.e., the corresponding morphism $T \to T'$ (see 2.2.13) is an isomorphism. Then the center of $\mathcal{W}(T)$ equals to ω_X. Let A be any chiral algebra. The tensor product $A \otimes \mathcal{W}(T)$ (see 3.5.15) contains A and $\mathcal{W}(T)$ as subalgebras, and A coincides with the centralizer of $T^\flat \subset \mathcal{W}(T) \subset A \otimes \mathcal{W}(T)$. We have a pair of adjoint functors

$$\mathcal{M}(X,A) \to \mathcal{M}(X, A \otimes \mathcal{W}(T)), \quad \mathcal{M}(X, A \otimes \mathcal{W}(T)) \to \mathcal{M}(X,A), \tag{3.8.2.1}$$

where the left functor is $M \mapsto M \otimes \mathcal{W}(T)$, the right one sends $N \in \mathcal{M}(X, A \otimes \mathcal{W}(T))$ to the centralizer of T.

[168] As always, "skew-symmetric" refers to the "super" tensor structure: if T is purely odd this amounts to "naive symmetric".

3.8.3 Lemma. The functor $\mathcal{M}(X,A) \to \mathcal{M}(X, A \otimes \mathcal{W}(T))$ is fully faithful. Its image consists of those $A \otimes \mathcal{W}(T)$-modules on which the Lie* algebra $T^\flat$ acts in a locally nilpotent way. In particular, if T is purely odd, then 3.8.2.1 are mutually inverse equivalences of categories. □

3.8.4. Let us return to situation 3.8.1 (so we drop the non-degeneracy condition). Assume that we have a direct sum decomposition $T = V \oplus V'$ such that $<>_T$ vanishes on both V and V'. Let $<> \in P_2^*(\{V', V\}, \omega_X)$ be the restriction of $<>_T$ (notice that the sign of $<>$ depends on the order of V, V'). One recovers $(T, <>_T)$ from $(V, V', <>)$ in the obvious way, so we will write $\mathcal{W} = \mathcal{W}(V, V', <>)$. One has the obvious embeddings of chiral algebras $\mathrm{Sym}V, \mathrm{Sym}V' \hookrightarrow \mathcal{W}(V, V')$.

Define a $\mathbb{Z}$-grading on $T^\flat$ setting $T^{\flat -1} := V$, $T^{\flat 1} := V'$, $T^{\flat 0} := \omega_X$. Therefore $\mathcal{W}$ acquires a $\mathbb{Z}$-grading.[169] One has $\mathcal{W}_1^{-1} = V$, $\mathcal{W}_1^1 = V'$, and $\mathrm{gr}_a \mathcal{W}^n = \mathrm{Sym}^{\frac{a-n}{2}} V \otimes \mathrm{Sym}^{\frac{a+n}{2}} V'$.

Remark. $\mathcal{W}$ carries another filtration – the one generated by $(\mathrm{Sym}V', V + \mathrm{Sym}V')$ (see 3.7.13). This filtration is commutative, and the associated graded algebra $\mathrm{gr}'\mathcal{W}$ equals $\mathrm{Sym}V' \otimes \mathrm{Sym}V$. So we have a canonical identification of graded algebras[170] $\mathrm{gr}'\mathcal{W} = \mathrm{Sym}(V' \oplus V) = \mathcal{W}^c$ which is compatible with coisson brackets. So if V, V' are mutually dual $\mathcal{D}_X$-vector bundles then $\mathcal{W}$ is a chiral $\mathrm{Sym}V'$-tdo (see 3.8.5 for terminology).

3.8.5 The Tate extension. Consider the $\mathcal{D}_X$-module $\mathcal{W}_2^0$. It contains $\mathcal{W}_0^0 = \mathcal{W}_0 = \omega_X$ and $\mathcal{W}_2^0/\omega_X = \mathrm{gr}_2 \mathcal{W}^0 = V \otimes V'$. The adjoint action of $\mathcal{W}_2^0$ preserves $\mathcal{W}_1^\pm$. Since $\omega_X \subset \mathcal{W}_2^0$ acts trivially, we get $*$ actions of $V \otimes V'$ on V, V'. Since $\mathcal{W}_2^0 = \mu(V, V') + \omega_X$ we see that $\mathcal{W}_2^0$ is a Lie* subalgebra of $\mathcal{W}$.

Recall that in 1.4.2 we defined the associative* algebra structure on $V \otimes V'$ together with its left action on V and right action on V'. We leave it to the reader to check that the above action of $V \otimes V'$ on V coincides with that of 1.4.2, the one on V' is opposite

[169] Notice that, unless V, V' are purely odd, the $\mathbb{Z}$-grading mod 2 does *not* coincide with the structure "super" $\mathbb{Z}/2$-grading.

[170] The grading we consider comes from the grading on $\mathcal{W}$; we forget about the grading coming from the filtration.

to that of 1.4.2, and the Lie* algebra structure on $V \otimes V' = \mathcal{W}_2^0/\omega_X$ coincides with the Lie* algebra structure associated with the associative* algebra structure from 1.4.2. So the quotient $V \otimes V' = \mathcal{W}_2^0/\omega_X$ is the Lie* algebra $\mathcal{G}^{Lie}$ from 2.6.2, and we have an extension of Lie* algebras

$$0 \to \omega_X \to \mathcal{W}_2^0 \to \mathcal{G} \to 0. \tag{3.8.5.1}$$

This extension canonically identifies with the extension $\mathcal{G}^{-\flat}$ opposite to the Tate extension $\mathcal{G}^{\flat}$ from 2.6.2. Namely, consider the chiral product $j_*j^*V \boxtimes V' \to \Delta_*\mathcal{W}_2^0$. Its restriction to $V \boxtimes V'$ equals to *minus* $<>$: $V \boxtimes V' \to \Delta_*\omega_X$, and its composition with the projection $\mathcal{W}_2^0 \to V \otimes V'$ is the standard projection $j_*j^*V \boxtimes V' \to \Delta_*(V \otimes V')$. Therefore it yields a canonical identification of extensions (see the construction of $\mathcal{G}^{\flat}$ in 2.6.2)

$$\mathcal{W}_2^0 \xrightarrow{\sim} \mathcal{G}^{-\flat}. \tag{3.8.5.2}$$

This identification is obviously compatible with the Lie* brackets.

3.8.6 Chiral Clifford algebras. For a vector space or a $\mathcal{D}$-module M set $M^{\Pi} := M \otimes k^{odd}$; this is a copy of M with reversed parity. Below we always use a canonical isomorphism $M^{\Pi} \otimes N^{\Pi} = M \otimes N$ defined as composition $(M \otimes k^{odd}) \otimes (N \otimes k^{odd}) = (M \otimes N) \otimes (k^{odd} \otimes k^{odd}) = M \otimes N$. We identify chiral or $*$ operations of arguments $\{M, N\}$ and $\{M^{\Pi}, N^{\Pi}\}$ in the similar way.

Let now Q be a $\mathcal{D}_X$-module without $\mathcal{O}$-torsion equipped with a *symmetric* $*$ pairing $(\,)_Q \in P_2^*(\{Q,Q\}, \omega_X)$. It amounts to a skew-symmetric $*$ pairing[171] $<>_{Q^{\Pi}}$ on Q^{Π}. Set $\mathcal{C}l = \mathcal{C}l(Q, (\,)_Q) := \mathcal{W}(Q^{\Pi}, <>_{Q^{\Pi}})$; this is the *Clifford* chiral algebra of $(Q, (\,)_Q)$.

Everything said in 3.8.4 has an obvious Clifford version, so we have a $\mathbb{Z}$-graded Clifford algebra $\mathcal{C}l^{\cdot}$ associated with a $*$ pairing $(\,) \in P_2^*(\{L', L\}, \omega_X) = P_2^*(\{L'[-1], L[1]\}, \omega_X)$ which contains commutative subalgebras $\mathrm{Sym}(L'[-1])$, $\mathrm{Sym}(L[1])$. The $*$ bracket of $\mathcal{C}l$ restricted to $L'[-1], L[1]$ equals to $(\,)$. On $\mathcal{C}l$ there is the standard filtration $\mathcal{C}l_{\cdot}$, and $\mathrm{gr}\mathcal{C}l = \mathrm{Sym}(L'[-1] \oplus L[1])$.[172]

[171] So $<>_{Q^{\Pi}}$ equals to $(\,)_Q$ on $Q^{ev} = Q^{\Pi odd}$ and to $-(\,)_Q$ on $Q^{odd} = Q^{\Pi even}$.

[172] See 1.1.16 for notation.

Let us pass to 3.8.5. Consider the Lie* algebra $\mathcal{C}l_2^0$. The chiral product $j_*j^*(L' \boxtimes L) = j_*j^*(L'[-1] \boxtimes L[1]) = j_*j^*(\mathcal{C}l_1^1 \boxtimes \mathcal{C}l_1^{-1}) \to \mathcal{C}l_2^0$ yields the $*$ pairing equal to $(\ )$, so it defines, as in 3.8.5, an isomorphism of ω_X-extensions of $\mathcal{G} = L \otimes L'$ (cf. 3.8.5.2)

$$\mathcal{C}l_2^0 \xrightarrow{\sim} \mathcal{G}^\flat \tag{3.8.6.1}$$

where $\mathcal{G}^\flat$ is the Tate extension defined by $L, L', (\ )$. This is an isomorphism of Lie* algebras, and it identifies the adjoint action of $\mathcal{C}l_2^0/\omega_X$ on $\mathcal{C}l_1^{\pm 1}$ with the usual action of $\mathcal{G}$ on $L'[-1]$, $L[1]$. The proof coincides with that of similar statement for Weyl algebras plus a sign exercise.

3.8.7. Let now L be a Lie* algebra on X which is a $\mathcal{D}_X$-vector bundle.[173] Let L' be the dual $\mathcal{D}_X$-vector bundle and $(\ ) \in P_2^*(\{L', L\}, \omega_X)$ the canonical non-degenerate pairing (see 2.2.13). As in 3.8.6 we have $T := L \oplus L'$ with the symmetric $*$ pairing $(\ )_T$, the $\mathbb{Z}$-graded Clifford algebra $\mathcal{C}l = \mathcal{C}l(T)$ and the Tate extension $\mathfrak{g}l(L)^\flat$ of Lie* algebra $\mathfrak{g}l(L) = L \otimes L'$ identified with $\mathcal{C}l(T)_2^0$. The adjoint action of L yields a morphism of Lie* algebras $L \to \mathfrak{g}l(L)$. Let $L^\flat$ be the pull-back to L of the extension $\mathfrak{g}l(L)^\flat$; this is *the Tate extension* of L. So we have a canonical morphism of Lie* algebras $\beta : L^\flat \to \mathfrak{g}l(L)^\flat \subset \mathcal{C}l$ which sends $1^\flat \in L^{\flat\ell}$ to $1 \in \mathcal{C}l^\ell$.

3.8.8. Let A be a chiral algebra and $\alpha : L^\flat \to A$ a morphism of Lie* algebras such that $\alpha(1^\flat) = -1_A$; such pair (A, α) is called *a BRST datum.* Since α and β take opposite values on $1^\flat \in L^\flat$ we have a morphism of Lie* algebras

$$\ell^0 := \alpha + \beta : L \to A \otimes \mathcal{C}l^0. \tag{3.8.8.1}$$

Let $L_\dagger$ be the DG Lie* algebra introduced in 1.4.5 (as a plain complex it equals $\mathcal{C}one(id_L)$). The above ℓ^0 extends to a morphism of graded Lie* algebras

$$\ell : L_\dagger \to A \otimes \mathcal{C}l, \tag{3.8.8.2}$$

where ℓ^{-1} is the composition of the embeddings $L[1] \to \mathcal{C}l^{-1} \to A \otimes \mathcal{C}l$.

[173] i.e., X-locally L is a projective $\mathcal{D}_X$-module of finite rank.

To define the BRST current we need a simple lemma. Let $\delta : L' \to \mathrm{Sym}^2(L'[-1])[2]$ be the Chevalley differential (see 1.4.9) and $i : \mathrm{Sym}(L'[-1]) \hookrightarrow \mathcal{C}l \hookrightarrow A \otimes \mathcal{C}l$ the usual embedding. As usual, $i^a : \mathrm{Sym}^n(L'[-1]) \hookrightarrow A \otimes \mathcal{C}l^n$ are the components of i.

3.8.9 Lemma. The $*$ operation $[\ell^0, i^1] + [\ell^{-1}, i^2\delta] \in P_2^*(\{L, L'\}, A \otimes \mathcal{C}l[1])$ vanishes. Here $[\ \ ]$ is the bracket on $A \otimes \mathcal{C}l$.

Proof. This statement has nothing to do with A. It follows immediately from the coincidence of $[\beta, i^1], -[\ell^{-1}, i^2\delta] \in P_2^*(\{L, L'\}, \mathcal{C}l[1])$ (now $[\ \ ]$ is the bracket in $\mathcal{C}l$): both operations equal to the coadjoint action of L composed with embedding i^1. □

Consider a chiral operation

$$\tilde{\chi} := \mu(\ell^0, i^1) + \mu(\ell^{-1}, i^2\delta) \in P_2^{ch}(\{L, L'\}, A \otimes \mathcal{C}l[1]) \tag{3.8.9.1}$$

where μ is the chiral product on $A \otimes \mathcal{C}l$. The corresponding $*$ operation vanishes by 3.8.9, so it amounts to a morphism of graded $\mathcal{D}_X$-modules

$$\chi : L \otimes L' \to A \otimes \mathcal{C}l[1]. \tag{3.8.9.2}$$

Since L is a $\mathcal{D}_X$-bundle we have $h(L \otimes L') \xrightarrow{\sim} \mathrm{End}(L)$. Set

$$\mathfrak{d} = \mathfrak{d}_{A,\alpha} := -\chi(id_L) \in \hbar(A \otimes \mathcal{C}l[1]) = h(A \otimes \mathcal{C}l^1)^{odd}. \tag{3.8.9.3}$$

This is *the BRST current* for A with respect to α. Its adjoint action is an odd derivation d of $A \otimes \mathcal{C}l$ of degree 1 is called *the BRST differential.*

3.8.10 Theorem. (i) One has $[\mathfrak{d}, \mathfrak{d}] = 0$, hence $d^2 = 0$.

The DG chiral algbebra $(A \otimes \mathcal{C}l, d)$ is denoted by $\mathcal{C}_{\mathrm{BRST}}(L, A)$.

(ii) ℓ commutes with differentials, so we have a morphism of DG Lie* algebras

$$\ell : L_\dagger \to \mathcal{C}_{\mathrm{BRST}}(L, A). \tag{3.8.10.1}$$

(iii) $\mathfrak{d}$ is a unique element of $\hbar(A \otimes \mathcal{C}l)$ such that $ad_{\mathfrak{d}}\ell^{-1} = \ell^0$. Similarly, d is a unique derivation of $A \otimes \mathcal{C}l$ of degree 1 such that $d\ell^{-1} = \ell^0$.

Remarks. (i) Property (ii) says that ℓ^{-1} is a d-homotopy between $\ell^0 = \alpha + \beta$ and 0; we refer to it as *the BRST property* of $\mathfrak{d}$ or d. According to (iii), the BRST property determines $\mathfrak{d}$ and d uniquely.[174]

(ii) The construction of the BRST element and the above theorem remain literally valid in the setting of chiral algebras on an algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ (see 3.3.9). We assume that L is a Lie* algebra on $\mathcal{Y}$ which is a $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$, etc.

Proof. It occupies 3.8.11–3.8.13.

3.8.11. First we show that $d\ell^{-1} = \ell^0 \in \mathrm{Hom}(L, A \otimes \mathcal{C}l^0)$, i.e., $\mathfrak{d}$ and d satisfy (iii).

Consider the operation $\mu(\tilde{\chi}, \ell^{-1}) \in P_3^{ch}(\{L, L', L\}A \otimes \mathcal{C}l)$. One has (in formulas below lower indices 1,2,3 label the variables) $\mu(\tilde{\chi}, \ell^{-1}) = \mu(\mu(\ell_1^0, i_2^1), \ell_3^{-1}) + \mu(\mu(\ell_1^{-1}, i^2\delta_2), \ell_3^{-1}) = \mu(\mu(\ell_1^0, \ell_3^{-1}), i_2^1) + \mu(\ell_1^0, \mu(i_2^1, \ell_3^{-1})) + \mu(i^2\delta_2, \mu(\ell_1^{-1}, \ell_3^{-1})) + \mu(\ell_1^{-1}, \mu(\ell_3^{-1}, i^2\delta_2))$. Let $[\tilde{\chi}, \ell^{-1}]$ be the restriction of this operation to $j_*j^*(L \boxtimes L') \boxtimes L \subset j_*^{(3)}j^{(3)*}(L \boxtimes L' \boxtimes L)$. One has $[\tilde{\chi}, \ell^{-1}] = \mu(\ell^{-1}[\ \ ]_{13}, i_2^1) + \mu(\ell_1^0, <>_{23}) + \mu(\ell_1^{-1}, [\ell_3^{-1}, i^2\delta_2])$. Notice that $[\ell^{-1}, i^2\delta] = -[\ell^0, i^1]$ since the ℓ-adjoint action of the graded Lie* algebra $L_\dagger$ on $\mathrm{Sym}(L'[-1]) \subset A \otimes \mathcal{C}l$ coincides with the canonical action from 1.4.9 which is compatible with the differentials. Thus $[\tilde{\chi}, \ell^{-1}] = -\mu(<>_{23}, \ell_1^0) - \mu([\ell_3^0, \ell_1^{-1}], i_2^1) - \mu(\ell_1^{-1}, [\ell_3^0, i_2^1]) = -\mu(<>_{23}, \ell_1^0) - [\mu(i_2^1, \ell_1^{-1}), \ell_3^0]$. The triple $*$ operations that correspond to both of the latter summands vanish, so we can rewrite them as binary $*$ operations $\gamma_1, \gamma_2 \in P_2^*(\{L \otimes L', L\}, A \otimes \mathcal{C}l)$, and our equality becomes $[\chi, \ell^{-1}] = -\gamma_1 - \gamma_2$. Thus $d\ell^{-1} = \gamma_1(id_L) + \gamma_2(id_L)$ where $\gamma_i(id_L) \in \mathrm{Hom}(L, A \otimes \mathcal{C}l)$ are morphisms obtained from γ_i by convolution with $id_L \in h(L \otimes L')$. One has $\gamma_1(id_L) = \ell^0$ and $\gamma_2(id_L) = 0$, and we are done.

3.8.12. Now let us prove the uniqueness statement of 3.8.10(iii).

(a) Let $F \subset h(A \otimes \mathcal{C}l)$ be the centralizer of ℓ^{-1}, so F contains $A \otimes \mathrm{Sym}(L[1])$. Let us show that actually $F = A \otimes \mathrm{Sym}(L[1])$. This statement follows from its "classical" version 1.4.21. Indeed, consider the ring filtration on $A \otimes \mathcal{C}l$ generated by $(A \otimes \mathrm{Sym}(L[1]), A \otimes (\mathrm{Sym}(L[1]) + L'))$ (see 3.7.13). The associated graded algebra equals $A \otimes \mathcal{C}l_c$ (see 1.4.20). Consider the corresponding filtration on $h(A \otimes \mathcal{C}l)$; we have $\mathrm{gr}h(A \otimes \mathcal{C}l) = h(A \otimes \mathcal{C}l_c)$. Take

[174] This is a chiral version of a theorem of Akman [Ak].

$v \in h(A \otimes \mathcal{C}l^a)$; let n be the smallest number such that it lies in nth term of the filtration, so the corresponding $\bar{v} \in \mathrm{gr}_n h(A \otimes \mathcal{C}l_c^a) = h(A \otimes \mathrm{Sym}^n(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1])$ is non-zero (cf. the proof of 1.4.21). Then the composition $ad_v \ell^{-1}$ takes values in the previous term, and the corresponding morphism from L^Π to $\mathrm{gr}_{n-1}(A \otimes \mathcal{C}l^a) \subset A \otimes \mathcal{C}l_c^{a-1}$ is the morphism $\psi(\bar{v})$ from 1.4.21. So 1.4.21 implies that for $v \in F$ one has $n = 0$, q.e.d.

(b) Consider the maps $h(A \otimes \mathcal{C}l^a) \to \mathrm{Der}(A \otimes \mathcal{C}l)^a \to \mathrm{Hom}(L^\Pi, A \otimes \mathcal{C}l^{a-1})$, the first arrow is the adjoint action, the second one is $\nu \mapsto \nu\ell^{-1}$. Now both of them are injective for $a \geq 1$; this obviously implies 3.8.10(iii). The proof is an obvious modification of that of its "classical" counterpart 1.4.23(iii) (replace 1.4.21 by (a) above and $\mathcal{C}l_c$ by $\mathcal{C}l$).

3.8.13. Let us show that $[\mathfrak{d}, \mathfrak{d}] = d(\mathfrak{d})$ vanishes. According to 3.8.12(a) it suffices to prove that $[\mathfrak{d}, \mathfrak{d}] \in F$, i.e., $d^2\ell^{-1} = 0$. One has $d\ell^{-1} = \ell^0$ (see 3.8.11), so we want to show that $d\ell^0 = 0$. Again by 3.8.12(a) it suffices to show that $d\ell^0$ takes values in F, i.e., $[d\ell^0, \ell^{-1}] = 0$. One has[175] $[d\ell^0, \ell^{-1}] = d[\ell^0, \ell^{-1}] - [\ell^0, d\ell^{-1}] = d\ell^{-1}[\quad] - [\ell^0, d\ell^{-1}] = \ell^0[\quad] - [\ell^0, \ell^0] = 0$, and we are done.

Finally 3.8.10(ii) follows from 3.8.11 and 3.8.13(a). □

3.8.14. The DG chiral algebra $\mathcal{C}_{\mathrm{BRST}}(L, A)$ considered as an object of an appropriate derived category is called *the BRST reduction* of A with respect to α. The reduction is *regular* if $H^a_{\mathrm{BRST}}(L, A) = 0$ for $a \neq 0$; then the BRST reduction is the plain chiral algebra $H^0_{\mathrm{BRST}}(L, A)$.

For any graded $A \otimes \mathcal{C}l$-module N the action of $\mathfrak{d}$ on N is a d-derivation of square 0, so N is automatically a DG $\mathcal{C}_{\mathrm{BRST}}(L, A)$-module. In particular, for an A-module M the module $M \otimes \mathcal{C}l^\cdot$ (see 3.8.2) is a DG $\mathcal{C}_{\mathrm{BRST}}(L, A)$-module. We denote it by $\mathcal{C}_{\mathrm{BRST}}(L, M)$ and call the BRST complex of L with coeficients in M. Its cohomology $H^\cdot_{\mathrm{BRST}}(L, M)$ is an $H^\cdot_{\mathrm{BRST}}(L, A)$-module; this is *BRST*, or *semi-infinite*, homology of L with coefficients in M. As follows from functoriality property (see 3.8.11(a)) $C_{\mathrm{BRST}}(L, M)$ depends, as a complex of plain $\mathcal{D}_X$-modules, only on the chiral L^b-module structure on M.

[175] The first equality comes since d is a derivation, the second and the fourth one follow since ℓ commutes with brackets, the third one is 3.8.11.

Variant. Let $x \in X$ be a point, $j_x : U_x \hookrightarrow X$ its complement. Assume that our BRST datum is given on U_x. According to 3.4.5, 3.5.17 the category of $j_{x*}A \otimes \mathcal{C}l$-modules supported at x coincides with the category of discrete $A_x^{as}\hat{\otimes}\mathcal{C}l_x^{as}$-modules. We can consider $\mathfrak{d}$ as an element of $h_x j_{x*}A \otimes \mathcal{C}l$, so its action yields a canonical differential on any discrete $A_x^{as}\hat{\otimes}\mathcal{C}l_x^{as}$-module. We will see in 3.8.25 that it can be described in terms of Tate linear algebra.

Remark. In this setting we have no canonical $j_{x*}\mathcal{C}l$-module supported at x, so to define the BRST homology of an A_x^{as}-module M we should choose some $\mathcal{C}l_x^{as}$-module C and then define the BRST homology of M as the homology of $M \otimes C$. If L is extended to X, i.e., we have a Lie* subalgebra $L_X \subset j_{x*}L$ then we can take for C the module corresponding to the fiber at x of the Clifford module for L_X.

3.8.15. Let us show that the BRST construction for coisson algebras from 1.4.20–1.4.24 can be considered as a "classical limit" of the chiral BRST.

Let A_t, $\alpha_t : L_t^\flat \to A_t$ be a $k[t]$-family of BRST data (see Remark (ii) in 3.8.10).[176] Assume that A_t is a flat $k[t]$-module, $A_0 := A/tA$ is a commutative chiral algebra, and the Lie* bracket $[\ \ ]_t$ on L_t is divisible by t.

The latter condition implies that the extension $L_0^\flat$ of $L_0 := L_t/tL_t$ splits canonically, so we have $\alpha_0 : L_0 \to A_0$. According to 3.2.6, A_0 is a coisson algebra. Set $[\ \ ]_t^- := t^{-1}[\ \ ]_t$; this is a Lie* bracket on L_t. Then α_0 is a morphism of Lie* algebras if we equip L_0 with bracket $[\ \ ]_0^- := [\ \ ]_t^- \mathrm{mod}t$. Thus we have the "classical" BRST complex $\mathcal{C}_{\mathrm{BRST}}(L_0, A_0)_c$ as defined in 1.4.23. We want to compare it with $\mathcal{C}_{\mathrm{BRST}}(L_t, A_t)$.

Let $\mathcal{C}l_t$ be the chiral Clifford $k[t]$-algebra for L_t, L_t', and $\mathcal{C}l_t^- \subset \mathcal{C}l_t$ the chiral subalgebra generated by $L[1]$ and $tL'[-1]$. Set $L_0 := L_t/tL_t$, $L_0' := L_t'/tL_t'$. We see that $\mathcal{C}l_0^- := \mathcal{C}l_t^-/t\mathcal{C}l_t^- = \mathrm{Sym}(L_0[1] \oplus L_0'[-1])$ is a commutative chiral algebra,[177] and the coisson bracket on $\mathcal{C}l_0^-$ that comes from the deformation $\mathcal{C}l_t^-$ is equal to the coisson Clifford bracket from 1.4.20. Therefore the coisson algebra $\mathcal{C}l_0^-$ coincides with the coisson Clifford algebra $\mathcal{C}l_c$ from 1.4.20.

[176] So L_t is a locally projective $\mathcal{D}_X[t]$-module, in particular, it is a $k[t]$-flat.

[177] Here the embedding $L_0'[-1] \hookrightarrow \mathcal{C}l_0^-$ comes from the the embedding $L_t'[-1] \hookrightarrow \mathcal{C}l_t$ multiplied by t.

Consider the graded chiral $k[t]$-subalgebra $A_t \otimes \mathcal{C}l_t^- \subset A_t \otimes \mathcal{C}l_t$. Its fiber over $t = 0$ equals $A_0 \otimes \mathcal{C}l_c$ (the tensor product of coisson algebras). The morphism $t\tilde{\chi}$ from (3.8.9.1) takes values in $A_t \otimes \mathcal{C}l_t^-$, so[178] we get a canonical element $\mathfrak{d}^- \in \hbar(A_t \otimes \mathcal{C}l_t^-[1])$ whose image in $\hbar(A_t \otimes \mathcal{C}l_t^-[1])$ equals $t\mathfrak{d}$. Set $\mathfrak{d}_c := \mathfrak{d}_0^- \in \hbar(A_0 \otimes \mathcal{C}l_c[1])$.

Since the algebra $A_0 \otimes \mathcal{C}l_0^-$ is commutative, the BRST differential $ad_{\mathfrak{d}}$ preserves the subalgebra $A_t \otimes \mathcal{C}l_t^- \subset A_t \otimes \mathcal{C}l$. We denote this DG $k[t]$-subalgebra of $\mathcal{C}(L_t, A_t)$ by $\mathcal{C}(L_t, A_t)^-$. As follows from the uniqueness statement 1.4.23(iii) its fiber at $t = 0$ coincides with the DG algebra $\mathcal{C}_{\mathrm{BRST}}(L_0, A_0)_c$ from 1.4.23. Notice that the differential on $\mathcal{C}(L_t, A_t)$ equals the adjoint action of $\mathfrak{d}^-$ with respect to the bracket $\{\quad\}_t$ from 3.2.6, so 1.4.23(iii) also shows that $\mathfrak{d}_c$ coincides with the "classical" BRST current of (1.4.22.2).

3.8.16 Example. Let A, $\alpha : L^\flat \to A$ be a BRST datum, and $A.$ be a commutative filtration on A (see 3.3.11) such that $\alpha(L^\flat) \subset A_1$. Define A_t as in Example from 3.2.6. Set $L_t := L \otimes k[t]$; let $\alpha_t : L_t \to A_t$ be the $k[t]$-linear morphism whose restriction to L equals the composition $L \to A_1 \subset A_t$. We equip L_t with a $k[t]$-linear Lie* bracket $[\quad]_t := t[\quad]_L$. Then (A_t, α_t) is a $k[t]$-family of BRST data that satisfies conditions of 3.8.15. So we have the corresponding flat $k[t]$-family $\mathcal{C}_{\mathrm{BRST}}(L_t, A_t)^-$ of DG chiral algebras whose fiber at $t = 1$ equals $\mathcal{C}_{\mathrm{BRST}}(L, A)$ and at $t = 0$ equals $\mathcal{C}_{\mathrm{BRST}}(L, \mathrm{gr}A)_c$.[179]

One can also look at the above situation as follows. Consider an increasing filtration on $A \otimes \mathcal{C}l$ with nth term equal to $\Sigma A_{n-i} \otimes \mathrm{Sym}(L'[-1]) \cdot \mathrm{Sym}^i(L[1])$. This is a commutative filtration; one has $\mathrm{gr}(A \otimes \mathcal{C}l) = \mathrm{gr}A \otimes \mathcal{C}l_c$ as coisson algebras. The BRST differential preserves our filtration. The classical BRST differential preserves the similar filtration[180] on $\mathrm{gr}A \otimes \mathcal{C}l_c$, and the above identification is compatible with the differentials: indeed, both associated graded algebras equal to the Chevalley complexes $\mathcal{C}(L, \mathrm{gr}A \otimes \mathrm{Sym}(L[1]))$. The above considerations show that the first terms of the corresponding spectral sequences also coincide. In particular, if the "classical" $E_1^{p,q}$ vanishes for $p + q \neq 0$ then both "classical" and "quantum" BRST reductions are regular.

[178] Replacing in (3.8.9.2), (3.8.9.3) χ by $t\chi$ and $\mathcal{C}l_t$ by $\mathcal{C}l_t^-$.
[179] With respect to the morphism $\alpha_0 : L \to \mathrm{gr}A$ equal to the composition $L \to A_1 \to \mathrm{gr}_1 A$.
[180] Which comes from the obvious grading.

In the rest of this section we will show that the associative topological DG algebras $\mathcal{C}_{\mathrm{BRST}}(L,A)^{as}_x$ can be computed purely in terms of Tate's linear algebra.

3.8.17 BRST in the Tate linear algebra setting. We assume that the reader knows the contents of 2.6.9–2.6.11. Below "topological vector space" means (unless stated explicitly otherwise) k-vector space equipped with a complete and separated linear topology. Topological vector spaces form a tensor category with respect to completed tensor product $\hat{\otimes}$.[181] If T is a Tate vector space, R any topological vector space then $R\hat{\otimes}T^*$ is equal to the space $\mathrm{Hom}(T,R)$ of continuous linear maps $T\to R$. More generally, $R\hat{\otimes}T^{\hat{\otimes}n}$ is equal to the space of continuous polylinear maps $T\times ..\times T\to R$. All topological associative algebras are assumed to be unital, complete and separated and the topology should be formed by open left ideals. The category of topological associative algebras is a tensor category with respect to $\hat{\otimes}$. For a topological associative algebra A we denote by $A\mathrm{mod}$ the category of left discrete A-modules. This is an abelian k-category.

For a topological Lie algebra L its *topological enveloping algebra* $U(L)$ is a topological associative algebra equipped with a continuous morphism of Lie algebras $L\to U(L)$ which is universal in the obvious sense. Thus $U(L)$ is a completion of the plain enveloping algebra with respect to topology made of left ideals generated by open subspaces $V\subset L$. If the topology on L is generated by open Lie subalgebras then $U(L)$ satisfies Poincaré-Birkhoff-Witt theorem: the associated graded algebra $\mathrm{gr}U(L)$ for the standard filtration[182] equals the completed symmetric algebra $\mathrm{Sym}L$. This condition holds automatically if L is a Tate vector space.[183]

Remark. If $L^\flat$ is a central extension of L by k then we have the corresponding twisted topological enveloping algebra $U(L)^\flat$; this is a topological associative algebra equipped with a morphism of topological Lie algebras $L^\flat\to U(L)^\flat$ which sends $1^\flat\in k\subset L^\flat$ to $1_{U(L)^\flat}$ and is universal morphism with these properties. The PBW theorem for $U(L)^\flat$

[181] $A\hat{\otimes}B$ is the completion of $A\otimes B$ with respect to topology formed by subspaces $U\otimes B+A\otimes V$ where U,V are open subspaces in A,B.

[182] Its terms are closures of the terms of the standard filtration on the plain enveloping algebra.

[183] Proof: for every c-lattice $V\subset L$ its normalizer in L intersected with V is an open Lie subalgebra of L contained in V.

holds if the topology on $L^\flat$ is generated by open Lie subalgebras (and we assume that the topology induced on $k \subset L^\flat$ is discrete).

Let T be a Tate (super) vector space, $<>=<>_T$ a skew-symmetric (continuous) bilinear form on T. Consider $T^\flat := T \oplus k$ as a Lie algebra with the non-zero component of the bracket equal to $<>$. This is a central extension of T (considered as a commutative Lie algebra) by k. The twisted topological enveloping algebra $W = U(T)^\flat$ is called *the topological Weyl algebra* of $(T, <>)$.

Assume that $<>$ is non-degenerate. Then the category Wmod can be described as follows. Let $V \subset T$ be a c-lattice; then its orthogonal complement $V^\perp$ is also a c-lattice. Assume that the restriction of $<>$ to V is trivial, i.e., $V^\perp \supset V$. The restriction of $<>$ to $V^\perp$ comes from the finite-dimensional vector space $V^\perp/V$. The Weyl algebra $W(V^\perp)$ is equal to the subalgebra of W generated by $V^\perp$ (and coincides with the centralizer of V in W). The projection $W(V^\perp) \to W(V^\perp/V)$ yields the functor $W(V^\perp/V)\text{mod} \to W\text{mod}$, $M \mapsto W \underset{W(V^\perp)}{\otimes} M$. This functor is fully faithful. According to a variant of Kashiwara's lemma it identifies $W(V^\perp/V)$mod with the subcategory of those W-modules on which every $v \in V \subset W$ acts in a locally nilpotent way. Every discrete W-module is a union of submodules which satisfy this property with respect to smaller and smaller V's.

As in 3.8.6 we can replace skew-symmetric pairing by a symmetric one to get a *topological Clifford algebra.* As above, we are interested in the graded version of this construction, so we start with a Tate vector space F and consider the topological graded Clifford algebra $\mathcal{C}l$ generated by the graded (super) Tate vector space $F^*[-1] \oplus F[1]$ such that the commutator pairing between $F^*[-1]$ and $F[1]$ equals the canonical pairing $F^*[-1] \otimes F[1] = F^* \otimes F \to k \subset \mathcal{C}l$. Our $\mathcal{C}l$ contains commutative graded subalgebras[184] $\text{Sym}(F^*[-1])$, $\text{Sym}(F[1])$. It carries the standard filtration $\mathcal{C}l.$ compatible with the grading; the algebra $\text{gr}\mathcal{C}l$ is equal to $\text{Sym}(F^*[-1] \oplus F[1])$.

We have $\mathcal{C}l_0^0 = k$, $\mathcal{C}l_1^1 = F^*[-1]$ and $\mathcal{C}l_1^{-1} = F[1]$. Notice that $\mathcal{C}l_2^0$ is equal to the normalizer of $F^*[-1] \oplus F[1]$ in $\mathcal{C}l$. The adjoint action of this Lie subalgebra preserves both

[184] Here Sym denotes the completed symmetric algebra.

$F^*[-1]$ and $F[1]$, and identifies $\mathcal{C}l_2^0/k$ with $\mathfrak{g}l(F)$ (as a topological Lie algebra).[185] Thus $\mathcal{C}l_2^0$ is a central extension of $\mathfrak{g}l(F)$ by k. It equals the Tate extension $\mathfrak{g}l(F)^\flat$ from 2.6.10:

3.8.18 Proposition. There is a canonical isomorphism of topological central extensions

$$\mathcal{C}l_2^0 \xrightarrow{\sim} \mathfrak{g}l(F)^\flat.$$

Proof. According to 2.6.1(ii) and 2.6.10 such an isomorphism amounts to a pair of continuous sections $s_\pm : \mathfrak{g}l_\pm(F) \to \mathcal{C}l_2^0$ such that $s_\pm$ commute with the adjoint action of $\mathfrak{g}l(F)$, $s_+ + s_- : \mathfrak{g}l_+(F) \oplus \mathfrak{g}l_-(F) \to \mathcal{C}l_2^0$ is an open map, and $s_+ - s_- : \mathfrak{g}l_0(F) \to k \subset \mathcal{C}l_2^0$ equals tr. Recall (see Exercise in 2.6.9) that $\mathfrak{g}l_\pm(F)$ are completions of $F \otimes F^*$ with respect to appropriate topologies. We define s_+, s_- restricted to $F \otimes F^*$ as, respectively, compositions $F \otimes F^* = F^*[-1] \otimes F[1] \xrightarrow{\cdot} \mathcal{C}l_2^0$, $F \otimes F^* = F[1] \otimes F^*[-1] \xrightarrow{\cdot} \mathcal{C}l_2^0$ where $\cdot$ is the product map. It follows immediately from Exercise in 2.6.9 that $s_\pm$ are continuous, so they extend to continuous maps $s_\pm : \mathfrak{g}l_\pm(F) \to \mathcal{C}l_2^0$. We leave it to the reader to check that $s_\pm$ satisfy the properties listed above. □

3.8.19. Let L be a topological Lie algebra which is a Tate vector space. The adjoint action yields a continuous morphism of Lie algebras $ad : L \to \mathfrak{g}l(L)$. The *Tate extension* of L is the pull-back $L^\flat$ of the Tate extension $\mathfrak{g}l(L)^\flat$ by ad.

Remark. For every Lie subalgebra $P \subset L$ which is a c-lattice the adjoint action maps P to $\mathfrak{g}l(L)_P$, so, according to Remark in 2.6.11, we have a canonical section $s_P : P \to L^\flat$. If $P' \subset P \subset L$ is another such subalgebra then the restriction of s_P to P' equals $s_{P'} + \mathrm{tr}_{P/P'}$ (the trace of the adjoint action of P' on P/P').

According to 3.8.18 we have a canonical morphism of topological Lie algebras $\beta : L^\flat \to \mathfrak{g}l(L)^\flat \subset \mathcal{C}l^0$ where $\mathcal{C}l$ is the Clifford algebra as in 3.8.17 (for $F = L$). One has $\beta(1^\flat) = 1_{\mathcal{C}l}$.

Let A be a topological associative algebra. A *BRST datum* is a continuous morphism of Lie algebras $\alpha : L^\flat \to A$ such that $\alpha(1^\flat) = -1_A$. Since α and β take opposite values on $1^\flat$ we have a morphism of topological Lie algebras

$$\ell^0 := \alpha + \beta : L \to A\hat{\otimes}\mathcal{C}l^0. \tag{3.8.19.1}$$

[185] Use Exercise in 2.6.9.

Let $L_\dagger$ be the topological DG Lie algebra with non-zero components $L_\dagger^0 = L, L_\dagger^{-1} = L[1]$, the differential $d = id_L$, and the Lie bracket on $L_\dagger^0$ equal to the bracket on L. The above ℓ^0 extends to a morphism of topological graded Lie algebras

$$\ell : L_\dagger \to A\hat{\otimes}\mathcal{C}l, \tag{3.8.19.2}$$

where ℓ^{-1} is the composition $L[1] \to \mathcal{C}l^{-1} \to A\hat{\otimes}\mathcal{C}l^{-1}$.

3.8.20. We are going to define *the BRST current* $\mathfrak{d}$ which is an odd element of $A\hat{\otimes}\mathcal{C}l^1$. Let $\mathfrak{d}_1^{\sim} \in A\hat{\otimes}\mathcal{C}l^0\hat{\otimes}L^*$ be the element that corresponds[186] to the linear map ℓ^0, and $\mathfrak{d}_1 \in A\hat{\otimes}\mathcal{C}l[1]^0$ the image of $\mathfrak{d}_1^{\sim}$ by the product map $\mathcal{C}l\hat{\otimes}L^*[-1] \to \mathcal{C}l$. Let $\mathfrak{d}_2^{\sim} \in L\hat{\otimes}L^*\hat{\otimes}L^*$ be the element that corresponds to the Lie bracket $L \times L \to L$, and $\mathfrak{d}_2 \in \mathcal{C}l[1]^0 \subset A\hat{\otimes}\mathcal{C}l[1]^0$ its image by the product map $L\hat{\otimes}L^*\hat{\otimes}L^* = (L[1]\hat{\otimes}L^*[-1]\hat{\otimes}L^*[-1])[1] \xrightarrow{\cdot} \mathcal{C}l[1]$. Set

$$\mathfrak{d} := \mathfrak{d}_1 - \frac{1}{2}\mathfrak{d}_2 \in A\hat{\otimes}\mathcal{C}l[1]^0. \tag{3.8.20.1}$$

The adjoint action of $\mathfrak{d}$ is *the BRST differential*. This is a continuous odd derivation of $A\hat{\otimes}\mathcal{C}l$ of degree 1.

3.8.21 Proposition. (i) One has $[\mathfrak{d}, \mathfrak{d}] = 0$, hence $d^2 = 0$.

The topological DG associative algbebra $(A \otimes \mathcal{C}l, d)$ is denoted by $\mathcal{C}_{\mathrm{BRST}}(L, A)$.

(ii) ℓ commutes with differentials, so we have a morphism of DG Lie* algebras

$$\ell : L_\dagger \to \mathcal{C}_{\mathrm{BRST}}(L, A). \tag{3.8.21.1}$$

This is *the BRST property* of $\mathfrak{d}$ or d. It determines $\mathfrak{d}$ and d uniquely:[187]

(iii) $\mathfrak{d}$ is a unique odd element in $A\hat{\otimes}\mathcal{C}l^1$ such that $ad_{\mathfrak{d}}\ell^{-1} = \ell^0$. Similarly, d is a unique continuous odd derivation of degree 1 of $A\hat{\otimes}\mathcal{C}l$ such that $d\ell^{-1} = \ell^0$.

Proof (cf.the proof of 3.8.10). (a) An immediate computation shows that $d\ell^{-1} = \ell^0$.

(b) It is clear that the centralizer F of $L[1] \subset \mathcal{C}l \subset A\hat{\otimes}\mathcal{C}l$ equals $A\hat{\otimes}\mathrm{Sym}(L[1]) \subset A\hat{\otimes}\mathcal{C}l$. In particular, F sits in degrees ≤ 0.

[186] See the beginning of 3.8.17.

[187] This is a version of a theorem of Akman [A].

(c) Let us check (iii). Property $[\mathfrak{d}, \ell^{-1}] = \ell^0$ determines $\mathfrak{d}$ uniquely up to an element of F^1. So $\mathfrak{d}$ is unique. Similarly, property $d\ell^{-1} = \ell^0$ determines d up to a derivation ∂ which kills the image of $L[1]$. Such ∂ sends $L^*[-1] \subset \mathcal{C}l^1 \subset A\hat{\otimes}\mathcal{C}l^1$ and $A \subset A\hat{\otimes}\mathcal{C}l^0$ to F. Since ∂ has degree 1 it kills $L^*[-1]$ and A according to (a). Thus $\partial = 0$, i.e., d is determined uniquely.

(d) (i) follows from (b); the argument repeats 3.8.13. (ii) follows from (i) and (a). □

3.8.22. For every graded discrete $A\hat{\otimes}\mathcal{C}l$-module N the action of $\mathfrak{d}$ on N is an odd d-derivation of degree 1 and square 0, so it defines on N a canonical structure of DG $\mathcal{C}_{\mathrm{BRST}}(L, A)$-module. Fix a graded discrete $\mathcal{C}l$-module C; then for a discrete A-module M the tensor product $M \otimes C$ becomes a DG $\mathcal{C}_{\mathrm{BRST}}(L, A)$-module called *the BRST, or semi-infinite homology,* complex of M (relative to C).

Remarks. (i) If L is purely even then an irreducible $\mathcal{C}l$-module is unique up to a twist by a 1-dimensional (super) vector space. So the BRST complex with respect to irreducible C is uniquely defined up to a twist by 1-dimensional graded (super) vector space.

(ii) BRST complexes behave in the obvious manner with respect to morphisms of A's. In particular, the BRST complex of M depends (as a complex) only on the $L^\flat$-module structure on M (coming from α).

3.8.23. Now we can compare the chiral and the Tate linear algebra versions of BRST. Let $x \in X$ be a (closed) point, $j_x : U_x \hookrightarrow X$ its complement.

Let L be a $\mathcal{D}$-vector bundle on U_x, L' the dual $\mathcal{D}$-vector bundle. According to 2.6.12 the Tate vector space $L'_{(x)} := \hat{h}_x(j_{x*}L')$ is dual to $L_{(x)}$, i.e., $L'_{(x)} = L^*_{(x)}$.

Let $\mathcal{C}l^{\cdot}$ be the graded chiral Clifford algebra generated by $L'[-1] \oplus L[1]$ (see 3.8.6), and $\mathcal{C}l^{\cdot}_{(x)}$ the topological graded Clifford algebra generated by $L'_{(x)}[-1] \oplus L_{(x)}[1]$ (see 3.8.17). According to 3.7.23 there is a canonical isomorphism of topological graded algebras $\mathcal{C}l_x^{\cdot as} \xrightarrow{\sim} \mathcal{C}l^{\cdot}_{(x)}$.

Here is a simple alternative construction of the identification $r^\flat$ of (2.6.16.1)[188] Consider the Tate extension $\mathfrak{g}l(L)^\flat = \mathcal{C}l_2^0$ (see (3.8.6.1)). The previous isomorphism yields then a

[188] Our L is V of 2.6.16.

morphism of Lie algebras $h_x j_{x*}\mathfrak{g}l(L)^\flat \to \mathcal{C}l^0_{(x)2}$. It is obviously continuous with respect to $\Xi^{sp\flat}_x$-topology (see 2.6.15), so we have a morphism of topological Lie $\mathfrak{g}l(L)^\flat_{(x)} \to \mathcal{C}l^0_{(x)2}$. Using the identification $\mathcal{C}l^0_{(x)2} = \mathfrak{g}l(L_{(x)})^\flat$ of 3.8.18 we can rewrite this morphism as $r^\flat : \mathfrak{g}l(L)^\flat_{(x)} \to \mathfrak{g}l(L_{(x)})^\flat$. By construction this $r^\flat$ is a lifting of the isomorphism $r : \mathfrak{g}l(L)_{(x)} \xrightarrow{\sim} \mathfrak{g}l(L_{(x)})$ of 2.6.14 to our central k-extensions.

Now we assume that our L is a Lie* algebra, so we have the Tate extension $L^\flat$ (see 3.8.7). Then $L_{(x)}$ is a topological Lie algebra (see 2.5.12, 2.5.13(ii)) and $L^\flat_{(x)}$ is a central extension of $L_{(x)}$ by k. By definition of $L^\flat$ the adjoint action of L lifts canonically to a morphism of topological extensions $L^\flat_{(x)} \to \mathfrak{g}l(L)^\flat_{(x)}$. Thus $r^\flat$ provides a canonical identification of $L^\flat_{(x)}$ with the Tate linear algebra extension $(L_{(x)})^\flat$ from 3.8.19.

Assume we have a chiral BRST datum, i.e., a chiral algebra A on U_x and a morphism of Lie* algebras $\alpha : L^\flat \to A$ such that $\alpha(1^\flat) = -1_A$ (see 3.8.8). This yields the topological associative algebra $A^{as}_x = (j_{x*}A)^{as}_x$ (see 3.4.2–3.4.12) and the morphism $\alpha_{(x)} : L^\flat_{(x)} \to A^{as}_x$ which sends $1^\flat$ to -1 which is a Tate linear algebra BRST datum (see 3.8.19).

According to 3.5.17 the functor $A \mapsto A^{as}_x$ commutes with tensor products, so $(A \otimes \mathcal{C}l)^{as}_x = A^{as}_x \hat{\otimes} \mathcal{C}l_{(x)}$.

3.8.24 Lemma. The canonical morphism (3.4.4.2) of graded Lie algebras

$$h_x j_{x*}A \otimes \mathcal{C}l \to (A \otimes \mathcal{C}l)^{as}_x = A^{as}_x \hat{\otimes} \mathcal{C}l_{(x)}$$

sends the chiral BRST current (3.8.9.3) to the Tate linear algebra one (3.8.20.1).

Proof. The image of chiral $\mathfrak{d}$ satisfies the BRST property. Now use 3.8.21(i). □

3.8.25. According to 3.4.5 the categories of graded $j_{x*}A \otimes \mathcal{C}l$-modules supported at x and discrete $A^{as}_x \hat{\otimes} \mathcal{C}l_{(x)}$-modules are canonically identified. Each of these objects carries a canonical BRST differential (defined as the action of the BRST currents), and the above lemma identifies these differentials. This provides a canonical identification of the BRST homology from 3.8.14 (Variant) with those from 3.8.22.

§9 Chiral differential operators

In this section we explain what are enveloping chiral algebras of Lie* algebroids, in particular chiral algebras of differential operators (case of the tangent algebroid). The key difference with the case of Lie* algebras treated in §7 is that the construction of enveloping algebra requires an extra structure (that of chiral extension) on the Lie* algebroid.

3.9.1. Let R be a commutative chiral algebra, so (see (3.3.3.1)) R^ℓ is a commutative $\mathcal{D}_X$-algebra.

We say that a chiral R-module M is an *R-differential module* or simply *$R_{\mathcal{D}if}$-module*, if M is a union of submodules which are successive extensions of commutative R-modules.[189] Equivalently, this means that there is a filtration $\{0\} = M_{-1} \subset M_0 \subset M_1 \subset .., \cup M_i = M$, such that $\mathrm{gr}_i M$ are commutative.[190] Denote by $\mathcal{M}(X,R)^{ch}_{\mathcal{D}if} \subset \mathcal{M}(X,R)^{ch}$ the full pseudo-tensor subcategory of $R_{\mathcal{D}if}$-modules. The subcategory $\mathcal{M}(X,R)_{\mathcal{D}if} \subset \mathcal{M}(X,R)$ is closed under direct sums and subquotients. Any chiral R-module N has the maximal differential submodule $N^{\mathcal{D}if}$. Every chiral operation preserves maximal differential submodules.

Consider the sheaf of commutative Lie algebras $h(R)$ on X. It acts canonically on every chiral R-module M, so M is a $\mathrm{Sym}h(R)$-module. If $M \in \mathcal{M}(X,R)$ is an $R_{\mathcal{D}if}$-module then the $h(R)$-action on M is locally nilpotent which means that every $m \in M$ is killed by some $\mathrm{Sym}^i h(R)$, $i >> 0$. The converse is true if we know that for every $i > 0$ the subsheaf of M that consists of section killed by $\mathrm{Sym}^i h(R)$ is $\mathcal{O}_X$-quasicoherent. This always happens if M is either $\mathcal{O}_X$-flat or supported in finitely many points.

Remark. Let $f : R_1^\ell \to R_2^\ell$ be a morphism of $\mathcal{D}_X$-algebras. If M is an $R_{2\mathcal{D}if}$-module, then it is an $R_{1\mathcal{D}if}$-module via f, so we have a faithful pseudo-tensor functor $f_! : \mathcal{M}(X,R_2)^{ch}_{\mathcal{D}if} \to \mathcal{M}(X,R_1)^{ch}_{\mathcal{D}if}$.

3.9.2. Consider the topological algebra with connection (R^{as}, ∇) defined in the end of 3.3. Since R is commutative R^{as} is also commutative. We have $R^\ell = R^{as}/I_R$ for an open ideal $I_R \subset R^{as}$. Let $R^{\mathcal{D}if}$ be the I_R-adic completion of R^{as}. Precisely, we consider on R^{as}

[189] See 3.3.6, 3.3.9. This definition makes sense for any (not necessary commutative) chiral algebra R.
[190] You may define M_i by induction taking for M_i the preimage of $(M/M_{i-1})^{com}$.

the topology defined by those open ideals $I \subset I_R$ that $I_R^n \subset I$ for $n >> 0$; our $R^{\mathcal{D}if}$ is the corresponding completion. This is a commutative topological $\mathcal{O}_X$-algebra equipped with a connection ∇.

We define discrete $R^{\mathcal{D}if}[\mathcal{D}_X]$-modules in the same way as we defined discrete $R^{as}[\mathcal{D}_X]$-modules in 3.4.6.[191] $R^{\mathcal{D}if}$*-polylinear* chiral operations between such modules are defined as in 3.4.9. So we have the pseudo-tensor category $\mathcal{M}^\ell(X, R^{\mathcal{D}if})^{ch}$.

The equivalence from 3.4.7 identifies $R_{\mathcal{D}if}$-modules with discrete $R^{\mathcal{D}if}[\mathcal{D}_X]$-modules. By 3.4.10 chiral operations between chiral $R_{\mathcal{D}if}$-modules are the same as $R^{\mathcal{D}if}$-polylinear chiral operations. Therefore we have the pseudo-tensor equivalence

$$\mathcal{M}(X, R)^{ch}_{\mathcal{D}if} \xrightarrow{\sim} \mathcal{M}^\ell(X, R^{\mathcal{D}if})^{ch}. \tag{3.9.2.1}$$

3.9.3 Proposition. (i) If $f : R_1^\ell \to R_2^\ell$ is an étale morphism of $\mathcal{D}_X$-algebras then the corresponding morphism of topological algebras $R_1^{\mathcal{D}if} \to R_2^{\mathcal{D}if}$ is also étale.

(ii) $R_{\mathcal{D}if}$-modules and chiral operations between them have étale local nature.

Proof. (i) Let $I_1 \subset R_1^{\mathcal{D}if}$ be an open ideal, $I_2 \subset R_2^{\mathcal{D}if}$ the closed ideal generated by $f(I_1)$. We want to show that I_2 is open and the morphism $R_1^{\mathcal{D}if}/I_1 \to R_2^{\mathcal{D}if}/I_2$ is étale. This follows from the fact that the morphism $R_1^{\mathcal{D}if} \to R_2^{\mathcal{D}if}$ is formally étale, which is an immediate corollary of 3.4.12.

(ii) Follows from (i), 3.9.2, and 3.4.10. □

Remark. The statement (ii) means, in particular, that for étale f the pseudo-tensor functor $f. : \mathcal{M}(X, R_2)^{ch}_{\mathcal{D}if} \to \mathcal{M}(X, R_1)^{ch}_{\mathcal{D}if}$ admits a left adjoint pseudo-tensor functor $f^\cdot : \mathcal{M}(X, R_1)^{ch}_{\mathcal{D}if} \to \mathcal{M}(X, R_2)^{ch}_{\mathcal{D}if}$. One has $f^\cdot M = R_2^{\mathcal{D}if} \underset{R_1^{\mathcal{D}if}}{\otimes} M$.

According to 3.9.3(ii) one has the notion of $\mathcal{O}_{\mathcal{D}if}$-module on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$; we also call these objects $\mathcal{O}$-*modules on* $\mathcal{Y}_{\mathcal{D}if}$. We denote the corresponding abelian pseudo-tensor category by $\mathcal{M}(\mathcal{Y})^{ch}_{\mathcal{D}if}$.

[191] Certainly discrete $R^{\mathcal{D}if}[\mathcal{D}_X]$-modules are the same as discrete $R^{as}[\mathcal{D}_X]$-modules such that the action of R^{as} comes from the discrete action of $R^{\mathcal{D}if}$.

3.9.4. For a commutative R a *chiral $R_{\mathcal{D}if}$-algebra* is a chiral algebra A together with a morphism of chiral algebras $R \to A$ such that A is an $R_{\mathcal{D}if}$-module. According to 3.9.3(ii) these objects have étale local nature, so we have the notion of chiral $\mathcal{O}_{\mathcal{D}if}$-algebra on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$; we also call them *chiral algebras on* $\mathcal{Y}_{\mathcal{D}if}$. The corresponding category is denoted by $\mathcal{CA}(\mathcal{Y})_{\mathcal{D}if}$.

Example. Let A be a chiral algebra equipped with a commutative filtration $R = A_0 \subset A_1 \subset ..$ (see 3.3.11). Then A is a chiral $R_{\mathcal{D}if}$-algebra. Notice that in this situation $\mathrm{gr}_1 A$ is a Lie* R-algebroid (see 1.4.10) in the obvious way.

Let A be a chiral $R_{\mathcal{D}if}$-algebra. We say that a chiral A-module M is *R-differential* if such is M considered as an R-module. This notion is étale local, so for any $\mathcal{Y}$ as above and $\mathcal{A} \in \mathcal{CA}(\mathcal{Y})_{\mathcal{D}if}$ we know what are *$\mathcal{O}$-differential $\mathcal{A}$-modules on $\mathcal{Y}$* or simply *A-modules on $\mathcal{Y}_{\mathcal{D}if}$*. These objects form an abelian pseudo-tensor category $\mathcal{M}(\mathcal{Y}, \mathcal{A})^{ch}_{\mathcal{D}if}$; we also have the corresponding sheaf of pseudo-tensor categories $\mathcal{M}(\mathcal{Y}_{\text{ét}}, \mathcal{A})^{ch}_{\mathcal{D}if}$ on $\mathcal{Y}_{\text{ét}}$.

3.9.5. Assume for the moment that R is smooth (see 2.3.16). A *chiral algebra of twisted differential operators on* Spec$R$ (we abbreviate it to chiral tdo on Spec$R$ or chiral R-tdo) is a chiral $R_{\mathcal{D}if}$-algebra $\mathcal{D}$ which admits a commutative filtration $\mathcal{D}_0 \subset \mathcal{D}_1 \subset ..$ (see 3.3.11) such that $R = \mathcal{D}_0$ and there is an isomorphism of coisson $\mathbb{Z}$-graded algebras $\mathrm{Sym}^{\cdot}\Theta_R \xrightarrow{\sim} \mathrm{gr}.\mathcal{D}$ identical on R.[192]

Notice that the filtration $\mathcal{D}.$ is uniquely defined: indeed, $\mathcal{D}_0 = R$, $\mathcal{D}_1$ is the normalizer of R in $\mathcal{D}$, and our filtration is 1-generated (see 3.7.13). We call it *the canonical filtration.* The above isomorphism of the coisson graded algebras is also uniquely defined.

It is clear that chiral tdo's are local objects for the étale topology. Thus we know what are chiral tdo's on any smooth algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. We denote the corresponding category by $\mathcal{TDO}(\mathcal{Y}) \subset \mathcal{CA}(\mathcal{Y})_{\mathcal{D}if}$. This is a groupoid. We have the corresponding sheaf of groupoids $\mathcal{TDO}(\mathcal{Y}_{\text{ét}})$ on $\mathcal{Y}_{\text{ét}}$.

The principal difference between chiral tdo's and tdo's in the usual setting (see, e.g., [BB]) is that there is no *canonical* ("non-twisted") chiral tdo. We also do not know if chiral

[192] Exercise: show that the existence of such an isomorphism amounts to the folowing conditions: the adjoint $*$ action of $\mathrm{gr}_1\mathcal{D}$ on $\mathcal{D}_0 = R$ identifies it with Θ_R and $\mathrm{gr}_1\mathcal{D}$ generates $\mathrm{gr}.\mathcal{D}$.

tdo's always exist locally in étale topology. This is true (as follows from (3.9.18.2)) if Θ_R is a locally trivial $\mathcal{D}$-vector bundle (which happens when R is a jet $\mathcal{D}_X$-algebra). If chiral tdo's exist locally, then they form a gerbe. We will see that this gerbe need not be trivial.

Just as in the usual setting one handles chiral tdo's realizing them as twisted enveloping algebras of certain Lie algebroids. We explain the chiral Lie algebroid basics in 3.9.6–3.9.14 returning to tdo's in 3.9.15.

3.9.6. Let R be any commutative chiral (super)algebra. An important class of chiral $R_{\mathcal{D}if}$-algebras is formed by chiral enveloping algebras of Lie* R-algebroids. It appears however that, contrary to the case of Lie* algebras, to define the enveloping algebra one needs an extra structure on our algebroid. We describe this structure first; for the definition of chiral enveloping algebra see 3.9.11.

Definition. Let $\mathcal{L}$ be a Lie* R-algebroid (see 1.4.10). A *chiral R-extension* of $\mathcal{L}$ is a $\mathcal{D}$-module extension $0 \to R \xrightarrow{i} \mathcal{L}^\flat \xrightarrow{\sigma} \mathcal{L} \to 0$ together with chiral R-module and Lie* algebra structures on $\mathcal{L}^\flat$ that satisfy the following compatibilities (a)–(d):

(a) i and σ are morphisms of chiral R-modules.

(b) σ is a morphism of Lie* algebras and i is compatible with the Lie* actions of $\mathcal{L}^\flat$.[193]

(c) The R-action $\mu_{\mathcal{L}^\flat} \in P_2^{ch}(\{R, \mathcal{L}^\flat\}, \mathcal{L}^\flat)$ is compatible with Lie* action of $\mathcal{L}^\flat$.[194]

(d) Consider the $*$-pairing $\mu^*_{\mathcal{L}^\flat} \in P_2^*(\{R, \mathcal{L}^\flat\}, \mathcal{L}^\flat)$ that comes from the chiral R-action $\mu_{\mathcal{L}^\flat}$. By (a) we know that it takes values in $i(R) \subset \mathcal{L}^\flat$ and vanishes when the second argument lies in $i(R)$. So $\mu^*_{\mathcal{L}^\flat} \in P_2^*(\{R, \mathcal{L}\}, R)$, and we demand that this operation equals to the structure $*$-action of $\mathcal{L}$ on R.

One calls a pair $(\mathcal{L}^\flat, \mathcal{L})$ as above[195] a *chiral Lie R-algebroid* or a *chiral Lie algebroid on* Spec$R$. These objects form a category $\mathcal{LA}^{ch}(R) = \mathcal{LA}^{ch}(\mathrm{Spec}R)$. By 2.5.15, 3.9.3 they have étale local nature, so we know what are chiral Lie algebroids on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. The corresponding category is denoted by $\mathcal{LA}^{ch}(\mathcal{Y})$, the sheaf of categories by $\mathcal{LA}^{ch}(\mathcal{Y}_{\text{ét}})$. For fixed $\mathcal{L}$ the groupoid of its chiral extensions is denoted by $\mathcal{P}^{ch}(\mathcal{L}) =$

[193] Here $\mathcal{L}^\flat$ acts on R via σ and the $\mathcal{L}$-action, and on $\mathcal{L}^\flat$ via the adjoint action.

[194] See 3.1.13(i).

[195] We will also denote it simply $\mathcal{L}^\flat$.

$\mathcal{P}^{ch}(\mathcal{Y},\mathcal{L})$, the corresponding sheaf of groupoids on $\mathcal{Y}_{\text{ét}}$ by $\mathcal{P}^{ch}(\mathcal{Y}_{\text{ét}},\mathcal{L})$.

Example. In the situation of Example in 3.9.4 A_1 is a chiral Lie R-algebroid. We will see in 3.9.13 that every $L^\flat$ that satisfies certain flatness condition comes in this way.

One usually constructs chiral extensions by means of the following lemma.

Let L be a Lie* algebra, τ an L-action on R, and $L^\flat$ an ω-extension of L (we follow notation of 3.7.20). For a chiral Lie R-algebroid $\mathcal{L}^\flat$ a *morphism* $(L^\flat,\tau)\to\mathcal{L}^\flat$ is a Lie* algebra morphism $\psi^\flat\colon L^\flat\to\mathcal{L}^\flat$ compatible with the actions on R and such that $\psi^\flat(1^\flat)=i(1_R)$. The set $\mathrm{Hom}((L^\flat,\tau),\mathcal{L}^\flat)$ of such morphisms depends on $\mathcal{L}^\flat$ in functorial way.

3.9.7 Lemma. If L is a flat $\mathcal{O}_X$-module[196] then the functor $\mathrm{Hom}((L^\flat,\tau),\cdot)$ is representable.

We denote the universal chiral Lie algebroid by $\mathcal{L}^\flat_R$.

Proof. One constructs $L^\flat_R$ as follows. Set $F:=R\boxtimes L^\flat+j_*j^*R\boxtimes 1^\flat\omega_X\subset j_*j^*R\boxtimes L^\flat$, so $j_*j^*R\boxtimes L^\flat/F=\Delta_*\mathcal{L}_R$ where $\mathcal{L}_R:=R^\ell\otimes L$. Let $\phi\colon F\twoheadrightarrow\Delta_*R$ be the morphism that equals τ on $R\boxtimes L^\flat$ and the canonical projection on $j_*j^*R\boxtimes\omega_X$. We define $\mathcal{L}^\flat_R$ as a plain $\mathcal{D}_X$-module by $\Delta_*L^\flat_R:=j_*j^*R\boxtimes L^\flat/\mathrm{Ker}\phi$, so we have the extension $0\to R\xrightarrow{i}L^\flat_R\xrightarrow{\sigma}L_R\to 0$.

Assume for a second that $R^\ell=\mathcal{O}_X$; then $\tau=0$ and $L^\flat_{\omega_X}=L^\flat$. Therefore[197] for any R the morphism $1_R:\omega_X\to R$ yields the canonical morphism $\psi^\flat:L^\flat\to L^\flat_R$. It is clear that $\psi:=\sigma\psi^\flat:L\to L_R$ is the obvious embedding $l\mapsto 1_R\otimes l$.

The chiral R-action on $L^\flat_R$ comes as follows. It is easy to see that the morphism $j_*j^*R\boxtimes L\to\Delta_*L^\flat_R$ is continuous with respect to the topology on $j_*j^*R\boxtimes L$ described in 3.4.8. Thus we may rewrite our construction saying that $L^\flat_R$ is the push-out of the extension $0\to I_R\otimes L\to R^{as}\otimes L\to R^\ell\otimes L\to 0$ by the morphism $\tau':I_R\otimes L\to R$ coming from the action τ. One easily checks that τ' is R^{as}-linear map, thus $L^\flat_R$ is a discrete $R^{as}[\mathcal{D}_X]$-module, i.e., an R-module (see 3.4.7, 3.4.8). □

As follows from the lemma (and its proof) for a chiral Lie algebroid $\mathcal{L}^\flat$ a morphism $\psi^\flat:(L^\flat,\tau)\to\mathcal{L}^\flat$ such that the corresponding morphism $\psi:L\to\mathcal{L}$ yields an isomorphism

[196] If R is $\mathcal{O}_X$-flat then this condition is irrelevant.

[197] Our construction if functorial with respect to compatible morphisms of R and $L^\flat$'s.

$R^\ell \otimes L \xrightarrow{\sim} \mathcal{L}$ amounts to an isomorphism $L^\flat_R \xrightarrow{\sim} \mathcal{L}^\flat$. Such $\psi^\flat$ is called $(L^\flat, \tau)$-*rigidification* of $\mathcal{L}^\flat$. If $L^\flat = L \oplus \omega_X$ is the trivialized extension then we call $\psi^\flat$ simply (L, τ)-*rigidification*.

3.9.8. It is important to compare chiral extensions with "classical" ones. By definition, a *classical R-extension* of a Lie* R-algebroid $\mathcal{L}$ is an extension $\mathcal{L}^c$ of $\mathcal{L}$ by R in the category of Lie* algebroids. So we have a $\mathcal{D}$-module extension $0 \to R \xrightarrow{i} \mathcal{L}^c \xrightarrow{\sigma} \mathcal{L} \to 0$ together with a Lie* R-algebroid structure on $\mathcal{L}^c$ such that σ is a morphism of Lie* algebroids and i is a morphism of R^ℓ-modules.[198] These objects have étale local nature, so, as above, we have the groupoid of classical extensions $\mathcal{P}^{cl}(\mathcal{L}) = \mathcal{P}^{cl}(\mathcal{Y}, \mathcal{L})$.

The Baer sum of classical extensions makes $\mathcal{P}^{cl}(\mathcal{L})$ a Picard groupoid. We can also form k-linear combinations of classical extensions, so $\mathcal{P}^{cl}(\mathcal{L})$ is a k vector space category.

If $\mathcal{L}^\flat$ is a chiral extension and $\mathcal{L}^c$ is a classical extension of $\mathcal{L}$ then we have the Baer sum of extensions $\mathcal{L}^c + \mathcal{L}^\flat$. This is a chiral R-module and a Lie* algebra in the obvious way. It obviously satisfies compatibilities (a)–(d) of 3.9.6, so $\mathcal{L}^c + \mathcal{L}^\flat$ is a chiral extension of $\mathcal{L}$. Therefore we have defined an Action of the Picard groupoid $\mathcal{P}^{cl}(\mathcal{L})$ on $\mathcal{P}^{ch}(\mathcal{L})$.

3.9.9 Lemma. If $\mathcal{P}^{ch}(\mathcal{L})$ is non-empty then it is a $\mathcal{P}^{cl}(\mathcal{L})$-Torsor in a canonical way.

Proof. For two chiral extensions their Baer difference is a classical extension. □

3.9.10. If $\mathcal{L}^\ell$ is a projective $R[\mathcal{D}_X]$-module then classical extensions of $\mathcal{L}$ may be described in terms of the Chevalley complex $C(\mathcal{L}) = C(\mathcal{L}, R)$ from 1.4.13 as follows. A $C(\mathcal{L})^{\geq 1}$-*torsor* is a $C(\mathcal{L})^1$-torsor S together with a d_C-affine map $d_S : S \to C(\mathcal{L})^2$ such that $d_C d_S = 0$. These objects form a Picard category $C(\mathcal{L})^{\geq 1}$-tors in the obvious way. Now there is a canonical equivalence of Picard categories

$$S : \mathcal{P}^{cl}(\mathcal{L}) \xrightarrow{\sim} C(\mathcal{L})^{\geq 1}\text{-tors} \tag{3.9.10.1}$$

which assigns to $\mathcal{L}^c$ the torsor $S(\mathcal{L}^c)$ of $R[\mathcal{D}_X]$-linear sections $s : \mathcal{L} \to \mathcal{L}^c$, $d_S(s) := [s, s] - s([\ \ ])$.

This description globalizes in the obvious manner. Assume that our $\mathcal{L}$ is a $\mathcal{D}_X$-bundle on $\mathcal{Y}$ (see 2.3.11). According to 2.5.16 we have the complex $\mathcal{C}(\mathcal{L})$ such that the complex of

[198] i automatically commutes with $\mathcal{L}^c$ actions.

sheaves $h(\mathcal{C}(\mathcal{L}))$ equals to the sheafified $C(\mathcal{L})$. We see that the group of automorphisms of a classical extension is[199] $H^1(Y_{\text{ét}}, DR(\mathcal{C}(\mathcal{L})^{\geq 1}))$, and the isomorphism classes of objects of $\mathcal{P}^{cl}(\mathcal{L})$ form the group $H^2(Y_{\text{ét}}, DR(\mathcal{C}(\mathcal{L})^{\geq 1}))$.

Assume in addition that locally on $\mathcal{Y}_{\text{ét}}$ our $\mathcal{L}$ admits a chiral extension. Then, as follows from 3.9.9, our $\mathcal{P}^{ch}(\mathcal{Y}_{\text{ét}}, \mathcal{L})$ is a $C(\mathcal{L})^{\geq 1}$-gerbe. So the obstruction to the existence of a global chiral extension lies in $H^3(Y_{\text{ét}}, DR(\mathcal{C}(\mathcal{L})^{\geq 1}))$.

3.9.11. Let $\mathcal{L}^\flat = (\mathcal{L}, \mathcal{L}^\flat)$ be a chiral Lie R-algebroid. For a chiral algebra A a *chiral morphism* $\mathcal{L}^\flat \to A$ is a morphism of $\mathcal{D}_X$-modules $\varphi : \mathcal{L}^\flat \to A$ compatible with all the structures. Precisely, this means that $\varphi i : R \to A$ is a morphism of chiral algebras and φ commutes with the chiral R-actions and $*$ brackets. The chiral morphisms $\mathcal{L}^\flat \to A$ depend on A in functorial manner, and 3.7.1 immediately implies that there is a universal morphism which we denote by $\mathcal{L}^\flat \to U(R, \mathcal{L})^\flat$. Indeed, $U(R, \mathcal{L})^\flat$ is a quotient of the chiral enveloping algebra[200] $U(\mathcal{L}^\flat)$ modulo the relations needed to make the canonical arrow $\mathcal{L}^\flat \to U(\mathcal{L}^\flat)$ a chiral morphism. We call $U(R, \mathcal{L})^\flat$ the *chiral envelope* of our chiral Lie algebroid.

It is clear that $U(R, \mathcal{L})^\flat$ is a chiral $R_{\mathcal{D}if}$-algebra and the enveloping algebra functor is compatible with étale localization of R. Therefore the above constructions make sense over any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. Notation: $U(\mathcal{L})^\flat_{\mathcal{Y}}$.

Exercise. For a chiral Lie algebroid $\mathcal{L}^\flat$ on $\mathcal{Y}$ and an $\mathcal{O}_{\mathcal{D}if}$-module M on $\mathcal{Y}$ define the notion of chiral action of $\mathcal{L}^\flat$ on M (see 3.7.16). Show that $\mathcal{O}_{\mathcal{D}if}$-modules on $\mathcal{Y}$ equipped with a chiral $\mathcal{L}^\flat$-action are the same as $U(\mathcal{L})^\flat_{\mathcal{Y}}$-modules on $\mathcal{Y}_{\mathcal{D}if}$ (cf. 3.7.18).

We define *the standard*, or *Poincaré-Birkhoff-Witt, filtration* on $U(R, \mathcal{L})^\flat$ as the image of the standard filtration on $U(\mathcal{L}^\flat)$. In other words, this is the filtration generated by the image of R and that of $\mathcal{L}^\flat$ (see 3.7.13). This filtration is commutative, so one has a canonical surjective morphism of $\mathbb{Z}$-graded coisson algebras

$$\text{Sym}^{\cdot}_{R^\ell} \mathcal{L}^\ell \twoheadrightarrow \text{gr.} U(R, \mathcal{L})^{\flat \ell}. \tag{3.9.11.1}$$

[199] Here DR is the de Rham complex along X.

[200] Here we consider $\mathcal{L}^\flat$ as a plain Lie* algebra.

One has the following version of the PBW theorem:

3.9.12 Theorem. If R is a flat $\mathcal{O}_X$-module and $\mathcal{L}$ is a flat R^ℓ-module then (3.9.11.1) is an isomorphism.

Proof. We deduce the statement from the ordinary PBW theorem 3.7.14. The reduction is of quite general nature. It also yields the PBW theorem for the usual Lie algebroids.[201] To make the argument more transparent we first present it in the setting of usual Lie algebroids (see 3.9.13), and then explain the slight modifications needed to cover the chiral situation (see 3.9.14).

The PBW theorem in the setting of usual Lie algebroids looks as follows. Let R be a commutative k-(super)algebra, L a Lie R-algebroid. Denote by $U(R,\mathcal{L})$ its enveloping algebra. This is an associative k-algebra equipped with a morphism of associative k-algebras $i : R \to U(R,\mathcal{L})$ and a morphism of Lie k-algebras $\varphi : \mathcal{L} \to U(R,\mathcal{L})$ such that φ is a morphism of R-modules (here R-acts on $U(R,\mathcal{L})$ via i followed by the left multiplication), and this datum is universal. It carries the standard, or PBW, filtration $U(R,\mathcal{L})_i$.[202] There is an obvious canonical morphism $\pi : \mathrm{Sym}^{\cdot}_R L \twoheadrightarrow \mathrm{gr}.U(R,\mathcal{L})$ of $\mathbb{Z}$-graded commutative (super)algebras. The PBW theorem says that

3.9.13 Theorem. π is an isomorphism if $\mathcal{L}$ is a flat R-module.

Proof. It consists of 3 steps A–C. First we consider the case when $\mathcal{L}$ admits a rigidification, i.e., $\mathcal{L} = R \otimes L$ for a Lie algebra L acting on R. Then our statement follows from the usual PBW for L. Next, in B we construct for any $\mathcal{L}$ a DG associative filtered superalgebra U together with an identification $H^0U = U(R,\mathcal{L})$ such that $\mathrm{gr}U$ is controllable (here step A is used). Finally, we show that if $\mathcal{L}$ is R-flat then U is quasiisomorphic to $U(R,\mathcal{L})$ as a filtered algebra which yields PBW.

A. Assume that $\mathcal{L}$ admits a rigidification, i.e., there exists a Lie k-algebra L and a morphism of Lie k-algebras $L \to \mathcal{L}$ which yields an isomorphism $R \otimes L \xrightarrow{\sim} \mathcal{L}$. Then the Lie

[201] Originally proved in [Rin] for Lie R-algebroids which are free R-modules.

[202] $U(R,\mathcal{L})_0$ is the image of R, $U(R,\mathcal{L})_1$ is the sum of the images of R and $\mathcal{L}$, and $U(R,\mathcal{L})_i = U(R,\mathcal{L})_1^i$ for $i > 0$.

algebra L acts on R. Denote by $U(R,L)$ the universal associative k-algebra equipped with an algebra morphism $i : R \to U(R,L)$ and a Lie algebra morphism $\varphi_L : L \to U(R,L)$ which are compatible with respect to this action.[203] We have the algebra morphism $R \to U(R,\mathcal{L})$ and the Lie algebra morphism $L \to U(R,\mathcal{L})$ which are compatible. It is clear that they satisfy the universality property, i.e., $U(R,L) = U(R,\mathcal{L})$. This leads to the following explicit description of $U(R,\mathcal{L})$.

The L action on R amounts to an action of the Hopf algebra $U(L)$ on the algebra R. Denote by $R \circledast U(L)$ the corresponding twisted tensor product of R and $U(L)$: this is an associative algebra which equals $R \otimes U(L)$ as a k-module with the product $(a \otimes u)\cdot(b \otimes v) = \Sigma a u_i(b) \otimes u_i' v$ where $\Sigma u_i \otimes u_i'$ is the coproduct of u. One checks in a moment that the obvious morphisms $R \to R \circledast U(L)$, $L \to R \circledast U(L)$ satisfy the above universality property, so we have a canonical isomorphism $U(R,L) \xrightarrow{\sim} R \circledast U(L)$. It identifies the standard filtration on $U(R,\mathcal{L})$ with the filtration $R \circledast U(L)_i := R \otimes U(L)_i$. Therefore the PBW theorem for $\mathcal{L}$ follows from the usual PBW for L.

B. Let $\mathcal{L}$ be *any* Lie R-algebroid. Let us define the DG associative filtered algebra U as promised above.

Consider $\mathcal{L}$ as a plain Lie algebra acting on R; let $\mathcal{L}_R = R \underset{k}{\otimes} \mathcal{L}$ be the corresponding rigidified Lie R-algebroid. We have the obvious surjective morphism of Lie R-algebroids $\mathcal{L}_R \twoheadrightarrow \mathcal{L}$; denote by K its kernel. Consider K as a plain R-module and set $\tilde{R} := \mathrm{Sym}_R(K[1])$; this is a $\mathbb{Z}$-graded commutative superalgebra. The Lie algebra $\mathcal{L}$ acts on $\tilde{R}$,[204] so we have the corresponding rigidified Lie $\tilde{R}$-algebroid $\mathcal{L}_{\tilde{R}} = \tilde{R} \underset{k}{\otimes} \mathcal{L}$ and its enveloping algebra $U := U(\tilde{R}, \mathcal{L}_{\tilde{R}})$. So we have the associative algebra morphism $\tilde{i} : \tilde{R} \to U$ and the Lie algebra morphism $\tilde{\varphi} : \mathcal{L}_{\tilde{R}} \to U$. As in step A let us consider $\tilde{i}$ together with the restriction $\tilde{\varphi}_{\mathcal{L}} : \mathcal{L} \to U$ of $\tilde{\varphi}$ to $\mathcal{L}$; these are morphisms of, respectively, associative and Lie algebras compatible via the $\mathcal{L}$-action on $\tilde{R}$, and U is the universal algebra equipped with such morphisms. The $\mathbb{Z}$-grading on $\tilde{R}$ yields $\mathbb{Z}$-gradings on $\mathcal{L}_{\tilde{R}}$ and U (we assume that $\mathcal{L}$ sits in degree 0).

[203] i.e., for $r \in R$, $l \in L$ one has $i(l(r)) = [\varphi_L(l) i(r)]$.

[204] The $\mathcal{L}$-action on $\tilde{R}^{-1} = K$ is the adjoint action of $\mathcal{L} \subset \mathcal{L}_R$ on $K \subset \mathcal{L}_R$.

We defined U as an enveloping algebra, so it carries the PBW filtration which is obviously compatible with the $\mathbb{Z}$-grading. Our U fits into the setting of A, so the corresponding associated graded algebra equals $\tilde{R} \underset{k}{\otimes} \mathrm{Sym}^{\cdot}_k \mathcal{L}$. We want to consider a different filtration: namely, the filtration which on a graded component U^a equals the PBW filtration shifted by a.[205] This is a ring filtration compatible with the grading; we call it *the canonical* filtration. Shifting appropriately the above PBW isomorphism we see that

$$\mathrm{gr}.U = \tilde{R}^{\cdot} \underset{k}{\otimes} \mathrm{Sym}^{\cdot}_k \mathcal{L} = \mathrm{Sym}^{\cdot}_R(K[1]) \underset{R}{\otimes} \mathrm{Sym}^{\cdot}_R \mathcal{L}_R = \mathrm{Sym}^{\cdot}_R(K[1] \oplus \mathcal{L}_R). \tag{3.9.13.1}$$

Our U carries a canonical derivation:

Sublemma. There is an odd derivation d of U of degree 1 which kills the images of R and $\mathcal{L}$ and such that on $\tilde{R}^{-1} = K[1]$ it equals the restriction ϕ of the canonical morphism $\mathcal{L}_{\tilde{R}} \xrightarrow{\varphi} U$ to $K \subset \mathcal{L}_R \subset \mathcal{L}_{\tilde{R}}$. Such d is unique.

Proof of Sublemma. The uniqueness of d is clear since our conditions define d on generators of U. It remains to show that d is correctly defined.

Set $\Lambda := \mathrm{Sym}(k[1]) = k \oplus k[1]$; this is a $\mathbb{Z}$-graded commutative (super)algebra. Consider the $\mathbb{Z}$-graded associative Λ-(super)algebra $U_\Lambda = U \otimes \Lambda = U \oplus U[1]$. We have the morphism of graded associative algebras $\tilde{R} \to U_\Lambda$ and that of Lie algebras $\mathcal{L} \to U_\Lambda$ which are compatible via the $\mathcal{L}$-action on $\tilde{R}$. They satisfy the obvious universality property.

As usual, an odd derivation of degree 1 of U is the same as an automorphism of the $\mathbb{Z}$-graded Λ-algebra U_Λ which induces identity on its quotient U. We will construct our d using the universality property of U_Λ.

The conditions on d mean that the corresponding automorphism α_d of U_Λ is identity on the images of R and $\mathcal{L}$, and on $\tilde{R}^{-1} = K[1]$ it equals $\nu := \tilde{i} + \phi : K[1] \to U \oplus U[1]$. Now ν takes values in the centralizer $Z_\Lambda \subset U_\Lambda$ of R; it is a morphism of R-modules, and it is compatible with $\mathcal{L}$-actions. Thus we get α_d from the universality property of U_Λ the moment we extend ν to a morphism of R-algebras $\tilde{R} = \mathrm{Sym}_R(K[1]) \to Z_\Lambda$. Such extension

[205] So the morphism $\tilde{i} : R \to U$ is compatible with filtrations if we equip R with the filtration coming from the grading.

is unique, and its existence amounts to the fact that on the image of ν the product of U_Λ is commutative. One checks this by a direct computation:

Namely, let ε, ε' be a generators of copies of $k[1]$ that have to do with $K[1]$ and Λ. Then for $a \in K$ one has $\nu(a \cdot \varepsilon) = \tilde{i}(a \cdot \varepsilon) + \phi(a)\varepsilon'$. Thus[206] $[\nu(a \cdot \varepsilon), \nu(b \cdot \varepsilon)] = [\phi(a)\varepsilon', \tilde{i}(b \cdot \varepsilon)] + [\tilde{i}(a \cdot \varepsilon), \phi(b)\varepsilon'] = (-1)^{\bar{b}+1}\, \tilde{i}([a,b] \cdot \varepsilon)\varepsilon' + (-1)^{\bar{b}}\, \tilde{i}([a,b] \cdot \varepsilon)\varepsilon' = 0$, q.e.d. □

It is clear that $d^2 = 0$ (check on the generators), so U is a DG algebra which sits in degrees ≤ 0. Our d is R-linear, and it preserves the canonical filtration. Its acts on $\mathrm{gr}_1 U = K[1] \oplus \mathcal{L}_R$ as the canonical embedding $K \to \mathcal{L}_R$, so we can rewrite (3.9.13.1) as a canonical isomorphism of graded DG algebras

$$\mathrm{gr}.U = \mathrm{Sym}^{\cdot}_R \mathcal{C}one(K \to \mathcal{L}_R). \tag{3.9.13.2}$$

The obvious morphisms $\tilde{R} \twoheadrightarrow R$, $\mathcal{L}_{\tilde{R}} \twoheadrightarrow \mathcal{L}$ yield a morphism of the enveloping algebras $U = U(\tilde{R}, \mathcal{L}_{\tilde{R}}) \to U(R, \mathcal{L})$. It identifies $U(R, \mathcal{L})$ (as a filtered algebra) with the quotient of U modulo the ideal generated by $\tilde{R}^{<0}$ and the image of $K \hookrightarrow \mathcal{L}_R \subset \mathcal{L}_{\tilde{R}}$. Therefore $U(R, \mathcal{L}) = H^0 U$.

C. Now assume that $\mathcal{L}$ is R-flat. Then $0 \to K \to \mathcal{L}_R \to \mathcal{L} \to 0$ is a short exact sequence of flat R-modules. Therefore the projection $\mathrm{Sym}^{\cdot}_R \mathcal{C}one(K \to \mathcal{L}) \twoheadrightarrow \mathrm{Sym}^{\cdot}_R \mathcal{L}$ is a quasiisomorphism. By (3.9.13.2) one has $H^a \mathrm{gr}.U = 0$ for $a \neq 0$, hence $H^0 \mathrm{gr}.U \xrightarrow{\sim} \mathrm{gr}.H^0 U$. By (3.9.13.2) one has $H^0 \mathrm{gr}.U = \mathrm{Sym}^{\cdot}_R \mathcal{L}$, so we proved the PBW theorem. □

3.9.14. *Proof of* 3.9.12. Let us return to the chiral setting. The above arguments should be modified as follows. Again, we proceed in three steps.

A. Assume that $\mathcal{L}^\flat$ admits a rigidification in the sense of 3.9.7, i.e., there exists a Lie* algebra L and a morphism of Lie* algebras $\psi^\flat : L \to \mathcal{L}^\flat$ such that $\psi := \sigma\psi^\flat : L \to \mathcal{L}$ yields an isomorphism $R^\ell \otimes L \xrightarrow{\sim} \mathcal{L}$. Then L acts on R via the action τ defined as the composition of ψ and the structure action of $\mathcal{L}$ on R.

We have the morphism of chiral algebras $i : R \to U(R, \mathcal{L})^\flat$ and the morphism of Lie* algebras $L \to U(R, \mathcal{L})^\flat$;[207] the morphism i is compatible with L-actions. One checks easily

[206] Here $\bar{a} \in \mathbb{Z}/2$ is the pairity of a.

[207] Defined as composition of $\psi^\flat$ and the structure morphism $\mathcal{L}^\flat \to U(R, \mathcal{L})^\flat$.

(cf. the proof of uniqueness in 3.9.7) that these morphisms satisfy universality property, i.e., $U(R,\mathcal{L})^\flat$ equals the algebra $R \circledast U(L)$ from 3.7.24.

Now 3.9.12 follows from the PBW result of 3.7.24.

Steps B,C literally repeat those of 3.9.13. □

3.9.15 Corollary. For smooth $\mathcal{Y}$ there is a canonical equivalence of groupoids (see 3.9.5)

$$\mathcal{TDO}^{ch}(\mathcal{Y}) \xrightarrow{\sim} \mathcal{P}^{ch}(\mathcal{Y}, \Theta_{\mathcal{Y}}).$$

Proof. We assign to a tdo $\mathcal{D}$ the term $\mathcal{D}_1$ of the canonical filtration with its obvious structure of the chiral algebroid. To a chiral algebroid $\Theta^\flat_{\mathcal{Y}}$ we assign its enveloping algebra $\mathcal{D}^\flat_{\mathcal{Y}} := U(\Theta_{\mathcal{Y}})^\flat_{\mathcal{Y}}$; this is a tdo by 3.9.12. □

The above results generalize to the DG setting in the obvious way. For example, for a DG commutative $\mathcal{D}_X$-superalgebra R[208] we know what are DG chiral $R_{\mathcal{D}if}$-algebras and DG chiral Lie R-algebroids. If R is smooth then we have the groupoid of chiral DG R-tdo's identified canonically with the groupoid of chiral DG R-extensions of Θ_R.

The following proposition is inspired by [MSV].[209] It may be considered as a version of 3.9.7 for L replaced by a single supersymmetry.

Let R^ℓ be a smooth $\mathcal{D}_X$-(super)algebra. Then $\Omega^\ell = \Omega_{R^\ell}$ is a smooth DG commutative $\mathcal{D}_X$-superalgebra.

3.9.16 Proposition. A DG Ω-tdo $\mathcal{D}_\Omega$ exists; it is unique up to a unique isomorphism.

Proof. (a) We need some facts about the structure of the DG Lie* Ω-algebroid Θ_Ω.

Set $\tilde{\Theta} = \mathcal{C}one(id_{\Theta_R})$, so $\tilde{\Theta}^{-1} = \tilde{\Theta}^0 = \Theta_R$. This complex carries a unique DG Lie* algebra structure such that the bracket on $\tilde{\Theta}^0$ is the usual Lie bracket on Θ_R.

Our $\tilde{\Theta}$ acts on Ω. The action morphism $\tau : \tilde{\Theta} \to \Theta_\Omega$ is constructed as follows. Notice that Ω sits in degrees ≥ 0 and Θ_Ω sits in degrees ≥ -1. Derivations in Θ_Ω^{-1} are R^ℓ-linear, and their action on Ω^1 yields an isomorphism $\Theta_\Omega^{-1} \xrightarrow{\sim} \mathcal{H}om_R(\Omega^1, R) = \Theta_R$. Its inverse is

[208] This means that R is a commutative $\mathcal{D}_X$-superalgebra equipped with an extra $\mathbb{Z}$-grading and an odd derivation of degree $+1$.

[209] The authors of [MSV] consider the case of a jet $\mathcal{D}$-scheme, and they work in the universal setting. We hope that in this situation our construction is equivalent to theirs.

a canonical embedding $\Theta_R[1] \hookrightarrow \Theta_\Omega$. Our τ is the morphism of complexes that extends this canonical embedding. One checks in a moment that $\tau^0 : \Theta_R \to \Theta^0_\Omega$ is the usual action of Θ_R on R extended to Ω by functoriality.[210] Hence τ is a morphism of DG Lie* algebras.

Let $\Theta_{\Omega/R} \subset \Theta_\Omega$ be the centralizer of $R^\ell \subset \Omega^\ell$; this is a graded Ω-submodule of Θ_Ω. The action of Θ_Ω on $R \subset \Omega$ yields the isomorphism $\Theta_\Omega/\Theta_{\Omega/R} \xrightarrow{\sim} \Omega^\ell \underset{R^\ell}{\otimes} \Theta_R$. The embedding $\tau^{-1} : \Theta_R[1] \hookrightarrow \Theta_{\Omega/R}$ extends to the isomorphism $\Omega^\ell \underset{R^\ell}{\otimes} \Theta_R[1] \xrightarrow{\sim} \Theta_{\Omega/R}$. Therefore we have the short exact sequence

$$0 \to \Omega^\ell \underset{R^\ell}{\otimes} \Theta_R[1] \to \Theta_\Omega \to \Omega^\ell \underset{R^\ell}{\otimes} \Theta_R \to 0 \tag{3.9.16.1}$$

of graded Ω-modules. The composition $\Theta_{\Omega/R} \xrightarrow{d} \Theta_\Omega \twoheadrightarrow \Theta_\Omega/\Theta_{\Omega/R}$ equals $id_{\Omega\otimes\Theta_R}$.

(b) *Uniqueness.* As follows from the DG version of 3.9.9, 3.9.15 the groupoid of DG Ω-tdo's is a Torsor over the category of classical DG Ω-extensions of Θ_Ω. The uniqueness amounts to triviality of the latter category.

So we want to prove that every such classical extension Θ^c_Ω admits a unique splitting. Since $\Omega^{-1} = 0$ we see that the projection σ yields the isomorphism $\Theta^{c-1}_\Omega \xrightarrow{\sim} \Theta^{-1}_\Omega$. Since Θ_Ω is generated as a DG Ω-module by Θ^{-1}_Ω we see that our splitting is unique.

Let us construct it. We see that τ lifts in a unique way to a morphism of complexes $\tau^c : \tilde{\Theta} \to \Theta^c_\Omega$. This is a morphism of Lie* algebras.[211] Let $\Theta'_\Omega \subset \Theta^c_\Omega$ be the Ω-submodule generated by the image of τ^c (i.e., the DG Ω-submodule generated by Θ^{c-1}_Ω). Now σ yields an isomorphism $\Theta'_\Omega \xrightarrow{\sim} \Theta_\Omega$. Indeed, τ^{c-1} is R^ℓ-linear, hence it yields a morphism $\Omega^\ell \underset{R^\ell}{\otimes} \Theta_R[1] \to \Theta'_\Omega$, and τ^{c0} is R^ℓ-linear modulo the image $\Theta'_{\Omega/R}$ of this morphism, so we have $\Omega^\ell \underset{R^\ell}{\otimes} \Theta_R \twoheadrightarrow \Theta'_\Omega/\Theta'_{\Omega/R}$; now use (3.9.16.1). The desired splitting is the inverse morphism; it is obviously compatible with all the structures.

(c) *Existence.* Consider the functor that assigns to a chiral DG algebra B the set of pairs (φ, ψ) where $\varphi: \Omega \to B$ is a morphism of chiral DG algebras and $\psi: \tilde{\Theta} \to B$ is a morphism

[210] It identifies Θ_R with $\mathrm{Ker} d : \Theta^0_\Omega \to \Theta^1_\Omega$.

[211] Take $\alpha, \beta \in \tilde{\Theta}^0$. Then α, β are exact, so $\alpha = d(\gamma)$ for $\gamma \in \tilde{\Theta}^{-1}$. One has $[\tau^c(\alpha), \tau^c(\beta)] = [d\tau^c(\gamma), \tau^c(\beta)] = d[\tau^c(\gamma), \tau^c(\beta)] = d\tau^c([\gamma, \beta]) = \tau^c d([\gamma, \beta]) = \tau^c([\alpha, \beta])$ (the third equality follows since the adjoint action of Θ^{c0}_Ω on $\Theta^{c-1}_\Omega = \Theta^{-1}_\Omega$ factors through Θ^0_Ω).

of graded Lie* algebras such that φ is compatible with $\tilde{\Theta}$-actions[212] and $\psi^{-1} : \Theta_R[1] \to B$ is R-linear.[213] This functor is representable by an appropriate quotient[214] $\mathcal{D}_\Omega$ of the chiral DG algebra $\Omega \circledast U(\tilde{\Theta})$. It remains to prove that $\mathcal{D}_\Omega$ is an Ω-tdo.

(d) Consider the DG Lie* Ω-algebroid $\tilde{\Theta}_\Omega := \Omega \underset{k}{\otimes} \tilde{\Theta}$ that corresponds to the action τ (see 1.4.12(i)). Let $K_\Omega \subset \tilde{\Theta}_\Omega$ be the kernel of the canonical morphism $\tau_\Omega : \tilde{\Theta}_\Omega \twoheadrightarrow \Theta_\Omega$. This is a DG Lie* Ω-subalgebra of $\tilde{\Theta}_\Omega$ that looks as follows.

Set $\tilde{\Theta}_{\Omega/R} := \Omega \underset{k}{\otimes} \tilde{\Theta}^{-1} = \Omega \otimes \Theta_R[1] \subset \tilde{\Theta}_\Omega$, $K_{\Omega/R} := K_\Omega \cap \tilde{\Theta}_{\Omega/R}$, and let K be the kernel of the product map $R^\ell \otimes \Theta_R \to \Theta_R$. One has $\tilde{\Theta}_\Omega^{-1} = R^\ell \otimes \Theta_R$ which yields identification $K_\Omega^{-1} = K$. It follows from (3.9.16.1) that $K_{\Omega/R} = \Omega^\ell \underset{R^\ell}{\otimes} K[1]$, we have a short exact sequence of graded Ω-modules $0 \to \Omega^\ell \underset{R^\ell}{\otimes} K[1] \to K_\Omega \to \Omega^\ell \underset{R^\ell}{\otimes} K \to 0$, and the composition $K_{\Omega/R} \xrightarrow{d} K_\Omega \twoheadrightarrow K_\Omega/K_{\Omega/R}$ is $id_{\Omega \otimes K}$.

(e) Let $\tilde{\Theta}_\Omega^\flat$ be the $(\tilde{\Theta}, \tau)$-rigidified chiral extension of $\tilde{\Theta}_\Omega$ (see 3.9.7). This is a DG chiral Lie Ω-algebroid. The projection σ yields the isomorphism $\tilde{\Theta}_\Omega^{\flat -1} \xrightarrow{\sim} \Theta_\Omega^{-1}$. Let $K_\Omega^\flat \subset \tilde{\Theta}_\Omega^\flat$ be the DG chiral Ω-submodule generated by $K_\Omega^{\flat -1} :=$ the preimage of K_Ω^{-1} in $\tilde{\Theta}_\Omega^{\flat -1}$. This is a Lie* ideal in $\tilde{\Theta}_\Omega^\flat$.[215]

Our $\mathcal{D}_\Omega$ is the quotient of $U(\Omega, \tilde{\Theta}_\Omega)^\flat$ modulo the ideal generated by the image of $K_\Omega^\flat$. We will prove in (f) that $i(\Omega) \cap K_\Omega^\flat = \mathrm{Ker}(K_\Omega^\flat \xrightarrow{\sigma} K_\Omega) = 0$. This implies that $\Theta_\Omega^\flat := \tilde{\Theta}_\Omega^\flat / K_\Omega^\flat$ is a chiral extension of Θ_Ω by Ω. So $\mathcal{D}_\Omega = U(\Omega, \Theta_\Omega)^\flat$ which is an Ω-tdo by 3.9.15, q.e.d.

(f) We want to show that the surjective map $K_\Omega^\flat \xrightarrow{\sigma} K_\Omega$ is an isomorphism. Our $K_\Omega^\flat$ lies in the centralizer of Ω,[216] so $K_\Omega^\flat$ is the Ω^ℓ-submodule generated by $K^{\flat -1} = K$ and $d(K^{\flat -1})$. Since $K^{\flat -1}$ is an R^ℓ-submodule, we have the map $\Omega^\ell \underset{R^\ell}{\otimes} K[1] \to K_\Omega^\flat$. This map is injective;[217] denote its image by $K_{\Omega/R}^\flat$. We have $\sigma : K_{\Omega/R}^\flat \xrightarrow{\sim} K_{\Omega/R}$. The composition $K = K_\Omega^{\flat -1} \xrightarrow{d} K_\Omega^{\flat 0} \twoheadrightarrow K_\Omega^{\flat 0}/K_{\Omega/R}^{\flat 0}$ is R^ℓ-linear, so it yields the morphism

---

[212] Here $\tilde{\Theta}$ acts on B via ψ and the adjoint action.

[213] Here R acts on B via $R \subset \Omega \xrightarrow{\varphi} B$.

[214] Imposed to assure R-linearity of ψ^{-1}.

[215] The Lie* subalgebra $i(\Omega) + \tilde{\Theta}_\Omega^{\flat -1} + d(\tilde{\Theta}_\Omega^{\flat -1})$ obviously normalizes $K_\Omega^\flat$ and generates $\tilde{\Theta}_\Omega^\flat$ as an Ω-module. Since the adjoint action of $K_\Omega^\flat$ on $\tilde{\Theta}_\Omega^\flat$ commutes with the chiral Ω-action we see that $\tilde{\Theta}_\Omega^\flat$ normalizes $K_\Omega^\flat$.

[216] Which equals the kernel of the projection $\tilde{\Theta}_\Omega^\flat \to \tilde{\Theta}_\Omega \to \Theta_\Omega$.

[217] Since such its composition with σ.

$\Omega^\ell \underset{R^\ell}{\otimes} K \to K^\flat_\Omega / K^\flat_{\Omega/R}$. This is an isomorphism,[218] hence $\sigma: K^\flat_\Omega / K^\flat_{\Omega/R} \xrightarrow{\sim} K_\Omega / K_{\Omega/R}$, and we are done. □

3.9.17. We will use $\mathcal{D}_\Omega$, or rather a part of it, to give a "classical" description of R-tdo's.

The part we need is $\mathcal{L}^\natural := \mathcal{D}^0_{\Omega 1} = \Theta^{\flat 0}_\Omega$.[219] This is a chiral Lie R-algebroid that can be described explicitly as follows.

Let $\Theta^\flat_{\Omega/R} \subset \Theta^\flat_\Omega$, $\mathcal{D}_{\Omega/R} \subset \mathcal{D}_\Omega$ be the centralizers of $R \subset \Omega$. Both $\Theta^{\flat -1}_\Omega = \Theta_R$ and $\Omega^1 \subset \Theta^{\flat 1}_\Omega$ are commutative Lie* subalgebras of $\Theta^\flat_{\Omega/R}$; the commutator between them takes values in $R \subset \Omega$ and it is equal to the canonical R-duality. These subalgebras generate $\mathcal{D}_{\Omega/R}$, so $\mathcal{D}_{\Omega/R}$ is the graded Clifford R-algebra generated by Θ_R and Ω^1_R. In particular, the image of the chiral pairing $j_* j^* \Theta_R \boxtimes \Omega^1_R \to \mathcal{D}^0_{\Omega 1} = \mathcal{L}^\natural$ is the Tate R-extension $\mathfrak{g}l(\Theta_R)^\flat$ of the Lie* R-algebra $\mathfrak{g}l(\Theta_R)$ (see ??). One checks immediately that this is the centralizer of R in $\mathcal{L}^\natural$. So we have the exact sequence $0 \to \mathfrak{g}l(\Theta_R)^\flat \to \mathcal{L}^\natural \xrightarrow{\sigma} \Theta_R \to 0$. There is a canonical Lie* algebra splitting $s^\natural : \Theta_R \to \mathcal{L}^\natural$ equal to the morphism ψ^0 from the part (c) of the proof of 3.9.16; so $s^\natural$ is the composition $\Theta_R \xrightarrow{\sim} \Theta^{\flat -1}_{\Omega 1} \xrightarrow{d} \Theta^{\flat 0}_{\Omega 1}$. The $s^\natural$-adjoint action of Θ_R on $\mathfrak{g}l(\Theta_R)^\flat$ coincides with the canonical action. The section $s^\natural$ is *not* R-linear: since $\Theta_R \xrightarrow{\sim} \Theta^{\flat -1}_{\Omega 1}$ is R-linear, the Leibnitz formula shows that the chiral operation $\mu_{\mathcal{L}^\natural}(id_R, s^\natural) - s^\natural \mu_{\Theta_R} \in P^{ch}_2(\{R, \Theta_R\}, \mathfrak{g}l(\Theta_R)^\flat)$ is equal to the composition of the Clifford pairing $\mu_{\mathcal{C}l} \in P^{ch}_2(\{\Omega^1, \Theta_R\}, \mathfrak{g}l(\Theta_R))$ with $d : R \to \Omega^1$.

Summing up: as a Lie* algebra $\mathcal{L}^\natural$ is the semi-direct product of Θ_R and $\mathfrak{g}l(\Theta_R)^\flat$ with respect to the canonical action of Θ_R on $\mathfrak{g}l(\Theta_R)^\flat$. The chiral action of R on $\mathcal{L}^\natural$ equals the obvious R^ℓ-action on $\Theta_R \oplus \mathfrak{g}l(\Theta_R)^\flat$ plus $\mu_{\mathcal{C}l}(d, id_{\Theta_R}) \in P^{ch}_2(\{R, \Theta_R\}, \mathfrak{g}l(\Theta_R))$.

We see that $\mathcal{L}^\natural$ is a chiral R-extension of the Lie* R-algebroid equal to the semi-direct product of Θ_R and $\mathfrak{g}l(\Theta_R)$. This Lie* R-algebroid identifies canonically with the Lie* R-algebroid $\mathcal{E}(\Theta_R)$ of infinitesimal symmetries of the $\mathcal{D}_X$-vector bundle Θ_R (see 2.3.16). Indeed, $\mathcal{E}(\Theta_R)$ is an extension of Θ_R by $\mathfrak{g}l(\Theta_R)$, and the Lie action of Θ_R on the $\mathcal{D}_X$-vector bundle Θ_R provides a Lie algebra section $s : \Theta_R \to \mathcal{E}(\Theta_R)$ which is compatible with the

[218] It is injective since such is the composition $\Omega^\ell \underset{R^\ell}{\otimes} K \to K^\flat_\Omega / K^\flat_{\Omega/R} \to K_\Omega / K_{\Omega/R}$.

[219] The upper index is the grading, the lower one is the canonical filtration. Here $\Theta^\flat_\Omega$ is the DG chiral Lie Ω-algebroid that corresponds to $\mathcal{D}_\Omega$.

R^ℓ-action exactly as $s^\natural \mathrm{mod} i(R)$ does.

3.9.18. Let now $\Theta_R^\flat$ be a chiral R-extension of Θ_R. Denote by $\mathcal{E}(\Theta_R)^\flat$ the extension of $\mathcal{E}(\Theta_R)$ by R defined as the Baer difference of $\mathcal{L}^\natural$ and the pull-back of the extension Θ_R by $\mathcal{E}(\Theta_R) \xrightarrow{\tau} \Theta_R$. So a section of $\mathcal{E}(\Theta_R)^\flat$ may be written as a pair (l, θ) where $l \in \mathcal{L}^\flat$, $\theta \in \Theta_R^\flat$ are sections such that the images of l, θ in Θ_R coincide, and $(l, \theta) = 0$ if and only if $\theta = i(r) = l$ for some $r \in R$.

The obvious chiral R-action on $\mathcal{E}(\Theta_R)^\flat$ is *commutative*, so $\mathcal{E}(\Theta_R)^\flat$ is an R^ℓ-module. It is also a Lie* algebra with bracket $[(l, \theta), (l', \theta')] = ([l, l'], [\theta, \theta'])$. So $\mathcal{E}(\Theta_R)^\flat$ is a Lie* R-algebroid which is an R-extension of $\mathcal{E}(\Theta_R)$. The kernel of the projection $\mathcal{E}(\Theta_R)^\flat \xrightarrow{\tau} \Theta_R$ is the Tate extension $\mathfrak{g}l(\Theta_R)^\flat$. Therefore $\mathcal{E}(\Theta_R)^\flat$ is a Tate structure on Θ_R (see 2.6.7).

We have defined a functor

$$\mathcal{P}^{ch}(\Theta_R) \to \mathcal{T}ate(\Theta_R), \quad \Theta_R^\flat \mapsto \mathcal{E}(\Theta_R)^\flat. \tag{3.9.18.1}$$

Both our categories are $\mathcal{P}^{cl}(\Theta_R)$-Torsors (see 3.9.9, 2.6.8). By construction our functor *anti*commutes with $\mathcal{P}^{cl}(\Theta_R)$-translations. In particular, it is an equivalence of groupoids. Using 3.9.15 we may rewrite it as a canonical equivalence

$$\mathcal{TDO}^{ch}(\mathcal{Y}) \xrightarrow{\sim} \mathcal{T}ate(\Theta_{\mathcal{Y}}) \tag{3.9.18.2}$$

valid for every smooth algebraic $\mathcal{D}_X$ space $\mathcal{Y}$.

§10 The universal setting

§11 Chiral homology

The chiral homology is a "quantum" version of the space of global horizontal sections of a $\mathcal{D}_X$-scheme.[220] One defines it as follows. We have seen in section 3.4 that chiral algebras amount to factorization algebras which are $\mathcal{D}$-modules on Ran's space $\mathcal{R}(X)$ (see 3.5.1). Now the chiral homology of X with coefficients in a chiral algebra is the de Rham homology of $\mathcal{R}(X)$ with coefficients in the corresponding $\mathcal{D}$-module.

[220] i.e., global solutions of a system of non-linear differential equations.

The section divides in three parts. First we make the above informal definition rigorous and establish some basic properties chiral homology (such as the compatibility with tensor and direct products); a key tool is the structure of Batalin-Vilkovisky algebra on the chiral homology complex. Then we "compute" chiral homology of chiral envelopes of Lie* algebras and algebroids, and explain the compatibility with the BRST reduction. Finally we show that when our picture varies in a way that the variation is governed by certain symmetries of chiral algebras then these symmetries yield a "connection" on the chiral homology bundles;[221] a typical example is the Knizhnik-Zamolodchikov equations.

3.11.1. We begin with motivational remarks.

The only problem with the above definition is that $\mathcal{R}(X)$ is *not* an ind-scheme in the strict sense[222] (recall that $\mathcal{R}(X)_i$ are not algebraic varieties for $i \geq 3$, see 3.5.2), so we are outside of the documented grounds of algebraic geometry. We know however that $\mathcal{R}(X)$ is an ind-scheme in a broad sense: it is an inductive limit of X^I's with respect to all diagonal embeddings (see 3.5.1(ii)). Luckily this inductive system satisfies an essential "cohomological descent" property (see lemma below) which permits to treat sheaves on $\mathcal{R}(X)$ in the same manner as if $\mathcal{R}(X)$ were an ind-scheme in the strict sense.

To make the sheaf-theoretic constructions clear in this subsection we explain them in plain topological setting of 3.5.1. What follows is basically an adaptation of the general formalism of SGA4 $\mathrm{V}^{\mathrm{bis}}$, except that we treat sheaves in a dual manner more suitable for the $\mathcal{D}$-module setting we have in mind. We consider just a few basic definitions; the more interesting subjects like duality and perverse sheaves are not touched.

So our X is any separated topological space. We have the topological space $\mathcal{R}(X)$; it is the union of the increasing sequence of closed subspaces $\mathcal{R}(X)_i$. On the other hand, we have an $\mathcal{S}^\circ$-diagram[223] $X^{\mathcal{S}}$, $I \mapsto X^I$, of topological spaces. The obvious projections $r_I : X^I \to \mathcal{R}(X)$ identify $\mathcal{R}(X)$ with the inductive limit of this diagram.

(i) We want to study sheaves on $\mathcal{R}(X)$ in terms of sheaves on $X^{\mathcal{S}}$. However, sheaves on

[221] A physicist would, probably, call it "Ward identities".

[222] i.e., $\mathcal{R}(X)$ is not a union of a directed system of closed subschemes.

[223] Recall that $\mathcal{S}$ is the category of finite non-empty sets and surjective maps.

a diagram of topological spaces have to do with sheaves on the *homotopy* direct limit of the diagram. We are saved by the next key lemma.

Denote by $\tilde{\mathcal{R}}(X)$ the homotopy inductive limit of our $\mathcal{S}^\circ$-diagram $X^{\mathcal{S}}$. By definition, this is the realization of the simplicial topological space whose l-simplices are pairs $(y, \tilde{n})$ where $\tilde{n}$ is an l-simplex of the classifying simplicial set $\mathcal{B}$ of $\mathcal{S}^\circ$ as defined in 2.2.19(iii), and $y \in X^{I_0}$. There is a canonical projection $\rho : \tilde{\mathcal{R}}(X) \to \mathcal{R}(X)$, $(y, \delta) \mapsto r_{I_0}(y)$.

In the above definition we have to replace $\mathcal{S}$ by a *small* category equivalent to it. Up to a canonical homotopy the choice of the small category is irrelevant. *From now on we replace* $\mathcal{S}$ *by a small category which we again denote by* $\mathcal{S}$ *by abuse of notation.*

Lemma. (i) The fibers of ρ sre contractible.

(ii) If X is a CW complex then ρ is a homotopy equivalence.

Proof. (i) For $v = [T] \in \mathcal{R}(X)$ the fiber $\rho^{-1}(v)$ is the realization of the simplicial set $B(\mathcal{S}/T)$ where $\mathcal{S}/T$ is the category whose objects are pairs (I, φ), $I \in \mathcal{S}$, $\varphi : I \twoheadrightarrow T$, the morphisms are the morphisms between I's compatible with φ's. This category is contractible since it has final object (T, id_T)).

(ii) Use [QK]??. □

Now let us pass to formal sheaf-theoretic definitions.

(ii) Let $\mathcal{I}$ be a small category and Y, $I \mapsto Y_I$, an $\mathcal{I}$-diagram of topological spaces.[224] We assume that every Y_I has finite cohomological dimension[225] (to be able to consider infinite complexes of sheaves without complications) and for each $\varphi : I \to J$ the map $\varphi_Y : Y_I \to Y_J$ is a closed embedding. A *!-sheaf on* Y is a rule P that assigns to each $I \in \mathcal{I}$ a sheaf of k-modules P_{Y_I} on Y_I and to each $\varphi : I \to J$ a morphism $\theta^{(\varphi)} : \varphi_{Y!} P_{Y_I} \to P_{Y_J}$. We demand that θ's are compatible with composition of φ's, and $\theta^{(id)}$ is the identity map. These objects form an abelian k-category category $\mathcal{S}h^!(Y)$. We denote by $D(Y)$ the corresponding derived category.

We say that a complex P of !-sheaves on Y is *admissible* if for each morphism $\varphi : I \to J$

[224] i.e., a functor on $\mathcal{I}$ with values in the category of topological spaces.

[225] i.e. there is $N = N_I < \infty$ such that for every open $U \subset Y_I$ and a sheaf F on U one has $H^{>N}(U, F) = 0$.

the morphism $\theta^{(\varphi)}$ yields a quasiisomorphism $P_{Y_I} \xrightarrow{\sim} R\varphi^!_Y P_{Y_J}$. Admissible complexes form a full triangulated subcategory $D_{adm}(Y) \subset D(Y)$.

A !-sheaf P on Y yields a functor ΓP on $\mathcal{I}$ with values in k-modules, $\Gamma P(I) := \Gamma(Y_I, P_{Y_I})$. Denote by $\Gamma(Y, P)$ the homotopy inductive limit $C(\mathcal{I}, \Gamma P)$ (see 3.12.1). The functor $\Gamma(Y, \cdot)$ admits a right derived functor: for a complex P of !-sheaves $R\Gamma(Y, P) = \Gamma(Y, \tilde{P})$ where $\tilde{P}$ is a flabby resolution of P.

(iii) We apply the above considerations to $\mathcal{I} = \mathcal{S}^\circ$ and the diagram $X^{\mathcal{S}}$, $I \mapsto X^I$. We assume that X is locally compact, separated and has finite cohomological dimension.

The sheaves on $\mathcal{R}(X)$ we will consider are sheaves of k-modules which are unions of subsheaves supported on $\mathcal{R}(X)_i$'s; they form a category $\mathcal{S}h(\mathcal{R}(X))$. The maps $r_I : X^I \to \mathcal{R}(X)$ are finite, so the push-forward functor $r_{I!} : \mathcal{S}h(X^I) \to \mathcal{S}h(\mathcal{R}(X))$ is exact. So it admits a right adjoint functor $r_I^! : \mathcal{S}h(\mathcal{R}(X)) \to \mathcal{S}h(X^I)$. We have a left exact functor $r^! : \mathcal{S}h(\mathcal{R}(X)) \to \mathcal{S}h^!(X^{\mathcal{S}})$, where $r^! F_{X^I} := r_I^! F$, and θ's are the obvious canonical morphisms.

Denote by $D(\mathcal{R}(X))$ the derived category of $\mathcal{S}h(\mathcal{R}(X))$. The functor $r^!$ admits the right derived functor $Rr^! : D(\mathcal{R}(X)) \to D(X^{\mathcal{S}})$. For a complex C of sheaves on $\mathcal{R}(X)$ one has $Rr^!(C) = r^!(\tilde{C})$ where $\tilde{C}$ is a flabby resolution of C. It is clear that $Rr^!(C)$ is an admissible complex.

The functor $r^!$ admits a left adjoint functor $r_! : \mathcal{S}h^!(X^{\mathcal{S}}) \to \mathcal{S}h(\mathcal{R}(X))$. Namely, a !-sheaf P on $X^{\mathcal{S}}$ defines a functor $P_{\mathcal{R}} : \mathcal{S}^\circ \to \mathcal{S}h(\mathcal{R}(X))$, $P_{\mathcal{R}}(I) := r_{I!} P_{X^I}$, $P_{\mathcal{R}}(\pi) := r_{J!}(\theta^{(\pi)})$, and $r_! P$ is the $\mathcal{S}^\circ$-inductive limit of $P_{\mathcal{R}}$.

The functor $r_!$ admits a left derived functor $Lr_! : D(X^{\mathcal{S}}) \to D(\mathcal{R}(X))$ which is left adjoint to $Rr^!$. It assigns to a complex P of !-sheaves on $X^{\mathcal{S}}$ the complex $C(\mathcal{S}^\circ, P_{\mathcal{R}})$. We leave it to the reader to define the adjunction morphisms and check the adjunction property.

Notice that $\Gamma(X^{\mathcal{S}}, P) = \Gamma(\mathcal{R}(X), C(\mathcal{S}^\circ, P_{\mathcal{R}}))$, so one has $R\Gamma(X^{\mathcal{S}}, P) = R\Gamma(\mathcal{R}(X), Lr_! P)$.

(iv) *The key fact:* The adjoint functors

$$D_{adm}(X^{\mathcal{S}}) \underset{Rr^!}{\overset{Lr_!}{\rightleftarrows}} D\mathcal{R}(X). \tag{3.11.1.1}$$

are mutually inverse equivalences of categories.

We leave the proof (which is an algebraic version of the lemma in 3.11.1(i)) as an exercise to the reader. One may also look at 3.11.4 below.

(v) For $n \geq 1$ let $\mathcal{S}_n \subset \mathcal{S}$ be the subcategory of sets of order $\leq n$. There is an obvious "truncated" version of all the above constructions with $\mathcal{S}$ replaced by $\mathcal{S}_n$ and $\mathcal{R}(X)$ replaced by $\mathcal{R}(X)_n$.

Denote by i_n the embedding $\mathcal{R}(X)_n \hookrightarrow \mathcal{R}(X)$, so we have the functors $i_n^!$, $i_{n!}$ between the categories of sheaves on $\mathcal{R}(X)_n$, $\mathcal{R}(X)$ and the corresponding derived functors. The $X^{\mathcal{S}}$ counterpart for these functors look as follows. The embedding $\mathcal{S}_n \hookrightarrow \mathcal{S}$ yields an obvious exact "truncation" functor $i_n^! : \mathcal{S}h^!(X^{\mathcal{S}}) \to \mathcal{S}h^!(X^{\mathcal{S}_n})$. It admits the left adjoint functor $i_{n!} : \mathcal{S}h^!(X^{\mathcal{S}_n}) \to \mathcal{S}h^!(X^{\mathcal{S}})$; for $Q \in \mathcal{S}h^!(X^{\mathcal{S}_n})$ and $I \in \mathcal{S}$ the sheaf $i_{n!}Q_{X^I}$ is the inductive limit of the sheaves $\Delta_*^{(I/T)}Q_{X^T}$, $T \in Q(I, \leq n)$ (:= the ordered set of quotients of I of order $\leq n$). The left derived functor $Li_{n!}$ has a similar description with the inductive limit replaced by the homotopy inductive limit. Both derived functors $Ri_n^!$ and $Li_{n!}$ send admissible complexes to admissible ones.

(vi) For every complex P of sheaves on $X^{\mathcal{S}}$ the complex $\Gamma(X^{\mathcal{S}}, P)$ has a canonical filtration $\Gamma(X^{\mathcal{S}}, P)_1 \subset \Gamma(X^{\mathcal{S}}, P)_2 \subset \ldots$ where $\Gamma(X^{\mathcal{S}}, P)_n := \Gamma(X^{\mathcal{S}_n}, i_n^! P) = C(\mathcal{S}_n^\circ, \Gamma P)$. Passing to the right derived functors we get a canonical filtration[226] on $R\Gamma(X^{\mathcal{S}}, P)$. If P is admissible then this filtration coincides with the filtration on $R\Gamma(\mathcal{R}(X), Lr_! P)$ coming from the filtration $\mathcal{R}(X)_n$ on $\mathcal{R}(X)$, so $\mathrm{gr}_n R\Gamma(X^{\mathcal{S}}, P) = R\Gamma(\mathcal{R}(X)_n^o, i_n^! Lr_! P)$.[227]

Summing up: we have identified the derived category of sheaves on $\mathcal{R}(X)$ with the derived category of admissible complexes of !-sheaves on $X^{\mathcal{S}}$. The cohomology functor $R\Gamma(\mathcal{R}(X), \cdot)$ is identified with the functor $R\Gamma(X^{\mathcal{S}}, \cdot)$.

[226] A filtration on a complex G, considered as an object of the derived category, is a filtered complex $G.$, considered as an object of the filtred derived category, together with an isomorphism (in the derived category) between G and the plain complex that corresponds to $G.$. The filtered derived category is formed by complexes F equipped with an increasing filtration $F.$ such that $F_a = 0$ for $a << 0$ and $\cup F_a = F$; the morphisms are morphisms of complexes preserving filtrations localized by filtered quasiisomorphisms (:= the morphisms that induce quasiisomorphisms between the successive quotients).

[227] Recall that $\mathcal{R}(X)_n^o$ is the locally closed stratum $\mathcal{R}(X)_n \smallsetminus \mathcal{R}(X)_{n-1}$.

3.11.2. Let us turn to the algebro-geometric setting. Now X is our curve. "Sheaf" means sheaf on the étale topology. The definition of !-sheaves on X^{S} from 3.11.1(ii) makes perfect sense in the present setting, so we have the abelian k-category $\mathcal{S}h^!(X^{\mathrm{S}})$, its derived category $D(X^{\mathrm{S}})$, the DG functor $\Gamma(X^{\mathrm{S}},\cdot)$ on the category of complexes $C(\mathcal{S}h^!(X^{\mathrm{S}})$, and its derived functor $R\Gamma(X^{\mathrm{S}},\cdot)$ together with the canonical filtration (3.11.1(vi)). The notion of admissible complex of !-sheaves from 3.11.1(ii) also makes sense in the present situation. In view of 3.11.1(iv) we denote the derived category of admissible complexes by $D(\mathcal{R}(X))$; admissible complexes will be also called complexes of sheaves on $\mathcal{R}(X)$. Thus $D(\mathcal{R}(X))$ is a full subcategory of $D(X^{\mathrm{S}})$. The restriction of the functor $R\Gamma(X^{\mathrm{S}},\cdot)$ to $D(\mathcal{R}(X))$ is denoted by $R\Gamma(\mathcal{R}(X),\cdot)$; the terms of its canonical filtration by $R\Gamma(\mathcal{R}(X),\cdot)_n = R\Gamma(\mathcal{R}(X)_n,\cdot)$.

3.11.3. Let us describe the successive quotients of the canonical filtration on $R\Gamma(\mathcal{R}(X),\cdot)$ as we did in 3.11.1(vi) for the topological setting.

For $n>0$ consider the space $\mathcal{R}(X)^o_n$ of configurations of n distinct points in X (see 3.5.1(i)). This is a smooth algebraic variety: we have a canonical projection $r_n : U^{(n)} \to \mathcal{R}(X)^o_n$ which identifies $\mathcal{R}(X)^o_n$ with the quotient of $U^{(n)}$ modulo the action of the symmetric group.[228]

Let P be a !-sheaf on X^{S}. The sheaf P_{X^n} is equivariant with respect to the action of the symmetric group, so its restriction to $U^{(n)}$ yields, by descent, the sheaf on $\mathcal{R}(X)^o_n$ which we denote by $P_{\mathcal{R}(X)^o_n}$. The functor $P \mapsto P_{\mathcal{R}(X)^o_n}$ is exact, so it extends to the derived categories in the obvious way.

Below we write $\Gamma(X^n,P) := \Gamma(X^n,P_{X^n})$, $\Gamma(\mathcal{R}(X)^o_n,P) := \Gamma(\mathcal{R}(X)^o_n,P_{\mathcal{R}(X)^o_n})$, etc.

Consider the canonical morphism of complexes $\Gamma(X^n,P) \to \Gamma(\mathcal{R}(X)^o_n,P)$ defined by the trace map $r_{n!}(P_{U^{(n)}}) \to P_{\mathcal{R}(X)^o_n}$. Composing it with the obvious projector[229] $\Gamma(X^{\mathrm{S}},P)_n = C(\mathcal{S}^\circ_n,\Gamma P) \to \Gamma(X^n,P)$ we get a morphism of complexes that vanishes on $\Gamma(X^{\mathrm{S}},P)_{n-1}$, i.e., a morphism $\mathrm{gr}_n\Gamma(X^{\mathrm{S}},P) \to \Gamma(\mathcal{R}(X)^o_n,P)$. Passing to derived functors (say, using flabby resolutions) we get a canonical morphism

$$\mathrm{gr}_n R\Gamma(X^{\mathrm{S}},P) \to R\Gamma(\mathcal{R}(X)^o_n,P). \tag{3.11.3.1}$$

[228] Recall that $U^{(n)}$ is the complement to the diagonal divisor in X^n.

[229] This projector is *not* compatible with the differential.

3.11.4 Lemma. If P is admissible then this arrow is a quasiisomorphism.

Proof. We assume that P is flabby. Consider a filtration $P.$ of P where P_a is a subcomplex whose sections over X^I are sections of P_{X^I} supported on diagonals of dimension $\leq a$. Notice that P_a, as well as P_a/P_b, $b \leq a$, are flabby admissible complexes. So our claim for P follows from the corresponding statements for $\mathrm{gr}_a P$'s. Therefore we may assume that $P = P_a$, $P_{a-1} = 0$.

The complexes $P_{\mathcal{R}(X)^o_n}$ vanish unless $n = a$. We have to consider 3 situations:

(i) $n < a$: Since $\Gamma(X^{\mathcal{S}}, P)_{a-1} = 0$ our claim is true for trivial reason.

(ii) $n = a$: Our $\Gamma(X^{\mathcal{S}}, P)_n$ is the chain complex of the symmetric group Σ_n acting on $\Gamma(X^n, P) = \Gamma(U^{(n)}, P)$. Since our complex is flabby the trace morphism $\mathrm{gr}_n \Gamma(X^{\mathcal{S}}, P) = \Gamma(X^{\mathcal{S}}, P)_n \to \Gamma(\mathcal{R}(X)^o_n, P)$ is a quasiisomorphism.

(iii) $n > a$: To prove our claim it suffices to show that $\Gamma(X^{\mathcal{S}}, P)_a \hookrightarrow \Gamma(X^{\mathcal{S}}, P)_n$ is a quasiisomorphism. We do it in 2 steps:

(a) Consider the complex P' of !-sheaves with $P'_{X^I} := \underset{T \in Q(I,a)}{\oplus} \Delta_.^{(I/T)} P_{X^T}$, the $\theta^{(\varphi)}$ are the obvious embeddings. There is a canonical map $P' \to P$ identical over X^a. Admissibility means that this is a quasiisomorphism, so we may assume that $P = P'$.

(b) Any Σ_a-module R defines a functor $\mathrm{Ind}R$ on $\mathcal{S}^\circ_n$, $I \mapsto \mathrm{Ind}R(I) := \underset{T \in Q(I,a)}{\oplus} R_T$ where R_T is the T-twisted copy of R. Consider the canonical map

$$C(\Sigma_a, R) = C(\mathcal{S}^\circ_a, \mathrm{Ind}R) \hookrightarrow C(\mathcal{S}^\circ_n, \mathrm{Ind}R). \tag{3.11.4.1}$$

Sublemma. This is a quasiisomorphism.

For $R = \Gamma(X^a, P)$ one has $\mathrm{Ind}R = \Gamma P$, so the sublemma implies our claim.

Proof of the sublemma. Denote by $\mathcal{Q}_n$ the category whose objects are pairs (I, T) where $I \in \mathcal{S}_n$, $T \in Q(I, a)$; the morphisms are surjections of I's that induce bijections between T's. Let $\tilde{\mathcal{Q}}_n$ be the category whose objects are pairs (I, φ), $I \in \mathcal{S}_n$, $\varphi : I \twoheadrightarrow \{1, .., a\}$, the morphisms are surjections of I's compatible with φ's. There are obvious functors $\tilde{\mathcal{Q}}_n \to \mathcal{Q}_n \to \mathcal{S}_n$; the group Σ_a acts on $\tilde{\mathcal{Q}}_n$. The projection $B\tilde{\mathcal{Q}}_n \to B\mathcal{Q}_n$ is a Σ_a-torsor; let $\tilde{R}$ be the local system on $B\mathcal{Q}_n$ defined by our Σ_a-module R and this covering. In other words, $\tilde{R}$ is the local system defined by functor $(I, T) \mapsto R_T$ on $\mathcal{Q}_n$.

Notice that $B\tilde{\mathcal{Q}}_n$ is contractible. So the embedding $B\Sigma_a = B\tilde{\mathcal{Q}}_a \subset B\tilde{\mathcal{Q}}_n$ yields a quasi-isomorphism of the chain complexes $C(\Sigma_a, R) \hookrightarrow C(B\tilde{\mathcal{Q}}_n, \tilde{R})$. The projection $\mathcal{Q}_n \to \mathcal{S}_n$ identifies the system of coefficients for $\mathrm{Ind}R$ with the push-forward of $\tilde{R}$. Therefore it yields an isomorphism of the chain complexes $C(B\tilde{\mathcal{Q}}_n, \tilde{R}) \xrightarrow{\sim} C(\mathcal{S}_n^\circ, \mathrm{Ind}R)$. The composition of these arrows is (3.11.4.1); we are done. □

3.11.5. So for $P \in D(\mathcal{R}(X))$ one has a canonical quasiisomorphism

$$\mathrm{gr}_n R\Gamma(\mathcal{R}(X), P) \xrightarrow{\sim} R\Gamma(\mathcal{R}(X)^o_n, P). \tag{3.11.5.1}$$

Consider the spectral sequence $E_r^{p,q}$ for our filtration. It converges to $H^\cdot(\mathcal{R}(X), P)$, and, by 3.11.4, one has

$$E_1^{p,q} = H^{p+q}(\mathcal{R}(X)^o_{-p}, P). \tag{3.11.5.2}$$

3.11.6. A *right $\mathcal{D}$-module M on $X^{\mathcal{S}}$* is a rule that assigns to $I \in \mathcal{S}$ a right $\mathcal{D}$-module M_{X^I} on X^I and to $\pi : J \twoheadrightarrow I$ a morphism of $\mathcal{D}$-modules $\theta^{(\pi)} : \Delta_*^{(\pi)} M_{X^I} \to M_{X^J}$ which satisfy same properties as θ's in the definition of !-sheaf. These objects form an abelian k-category $\mathcal{M}(X^{\mathcal{S}})$. A complex M of right $\mathcal{D}$-modules on $X^{\mathcal{S}}$ (a.c.a. *right $\mathcal{D}$-complex on $X^{\mathcal{S}}$*) is *admissible* if for each π the morphism $M_{X^I} \to R\Delta^{(\pi)!} M_{X^J}$ defined by $\theta^{(\pi)}$ is a quasiisomorphism. We also call an admissible right $\mathcal{D}$-complex on $X^{\mathcal{S}}$ a *right $\mathcal{D}$-complex on $\mathcal{R}(X)$*. The corresponding derived category is denoted by $D\mathcal{M}(\mathcal{R}(X))$. Below we skip word "right" when it is clear that we deal with right $\mathcal{D}$-modules.

An increasing filtration on a $\mathcal{D}$-complex M is *admissible* if $\mathrm{gr}_i M$ are admissible complexes. The corresponding filtered derived category is denoted by $DF\mathcal{M}(\mathcal{R}(X))$.

Notice that the above definitions make sense for any diagram of varieties and closed embeddings.

Remark. If L is a left $\mathcal{D}$-module on $\mathcal{R}(X)$ in the sense of 3.5.2 and M is a right $\mathcal{D}$-module on $X^{\mathcal{S}}$ then we have a right $\mathcal{D}$-module $L \otimes M$ on $X^{\mathcal{S}}$ such that $(L \otimes M)_{X^I} = L_{X^I} \otimes M_{X^I}$ and its structure morphisms $\theta^{(\pi)}$ come from the corresponding morphisms $\nu^{(\pi)}$ and $\theta^{(\pi)}$ for L and M via (2.1.3.1). If L is flat[230] and M is an admissible complex then the

[230] As an $\mathcal{O}_{\mathcal{R}(X)}$-module.

complex $L \otimes M$ is admissible. So the tensor category $\mathcal{M}^\ell(\mathcal{R}(X))$ acts on $\mathcal{M}(X^{\mathcal{S}})$, and the tensor DG category of complexes $C(\mathcal{M}^\ell(\mathcal{R}(X)))$ acts on $C(\mathcal{M}(X^{\mathcal{S}}))$. The action of the tensor subcategory of complexes of flat modules[231] preserves admissible complexes and quasiisomorphisms, so it yields an action on $D\mathcal{M}(\mathcal{R}(X))$.

A $\mathcal{D}$-complex M on $X^{\mathcal{S}}$ yields a complex $DR(M)$ of !-sheaves on $X^{\mathcal{S}}$, $DR(M)_{X^I} := DR(M_{X^I})$. The functor DR preserves quasiisomorphisms and sends admissible complexes to admissible ones,[232] so we have the exact functors $DR : D\mathcal{M}(X^{\mathcal{S}}) \to D(X^{\mathcal{S}})$, $D\mathcal{M}(\mathcal{R}(X)) \to D(\mathcal{R}(X))$. For a $\mathcal{D}$-complex M we set $\Gamma_{DR}(X^{\mathcal{S}}, M) := \Gamma(X^{\mathcal{S}}, DR(M))$ and $R\Gamma_{DR}(X^{\mathcal{S}}, M) := R\Gamma(X^{\mathcal{S}}, DR(M))$.[233] These complexes carry a canonical filtration coming from the canonical filtration on Γ and $R\Gamma$. For admissible M we write $H^i_{DR}(\mathcal{R}(X), M) := H^i R\Gamma_{DR}(\mathcal{R}(X), M)$, $H^i_{DR}(\mathcal{R}(X)_n, M) := H^i(R\Gamma_{DR}(\mathcal{R}(X), M)_n)$.

Repeating the construction from 3.11.3 we see that a $\mathcal{D}$-complex M on $X^{\mathcal{S}}$ defines a complex $M_{\mathcal{R}(X)^o_n}$ of $\mathcal{D}$-modules on every $\mathcal{R}(X)^o_n$. One has $DR(M_{\mathcal{R}(X)^o_n}) = DR(M)_{\mathcal{R}(X)^o_n}$, so, by 3.11.4, for $M \in D\mathcal{M}(\mathcal{R}(X))$ we get a canonical quasiisomorphism

$$\mathrm{gr}_n R\Gamma_{DR}(\mathcal{R}(X), M) \xrightarrow{\sim} R\Gamma_{DR}(\mathcal{R}(X)^o_n, M) \tag{3.11.6.1}$$

and the spectral sequence $E_r^{p,q}$ converging to $H^{\cdot}_{DR}(\mathcal{R}(X), M)$ with the first term

$$E_1^{p,q} = H^{p+q}_{DR}(\mathcal{R}(X)^o_{-p}, M). \tag{3.11.6.2}$$

3.11.7. There is an exact fully faithful functor

$$j^{(n)}_{\mathcal{R}*} : \mathcal{M}(\mathcal{R}(X)^o_n) \hookrightarrow \mathcal{M}(X^{\mathcal{S}}) \tag{3.11.7.1}$$

defined by $(j^{(n)}_{\mathcal{R}*}N)_{X^I} := \underset{T \in Q(I,n)}{\oplus} \Delta_*^{(I/T)} j_*^{(I)} N_{U^{(T)}}$, $N \in \mathcal{M}(\mathcal{R}(X)^o_n)$. Here $N_{U^{(T)}}$ is the pull-back of N by the étale projection $U^{(T)} \to \mathcal{R}(X)^o_n$. Note that $N = (j^{(n)}_{\mathcal{R}*}N)_{\mathcal{R}(X)^o_n}$.

A *Cousin $\mathcal{D}$-complex on* $\mathcal{R}(X)$ is a complex M of right $\mathcal{D}$-modules on $X^{\mathcal{S}}$ such that $M^{-n} \in j^{(n)}_{\mathcal{R}*}(\mathcal{M}(\mathcal{R}(X)^o_n))$ (in particular, $M^a = 0$ for $a \geq 0$). Such M is automatically admissible. Cousin $\mathcal{D}$-complexes form an abelian category $\mathcal{C}ous(\mathcal{R}(X))$.

[231] We can play with infinite length complexes without worries since the Tor dimension is finite.

[232] DR commutes with $\Delta^{(\pi)!}$, see 2.1.6.

[233] For an explicit way to compute $R\Gamma_{DR}$ see 3.12.6.

Remark. The functor $\mathcal{C}ous(\mathcal{R}(X)) \to D\mathcal{M}(\mathcal{R}(X))$ is a fully faithful embedding. In fact, $\mathcal{C}ous(\mathcal{R}(X))$ is the core of a certain canonical t-structure on $D\mathcal{M}(\mathcal{R}(X))$.

Let M be a Cousin $\mathcal{D}$-complex. According to (3.11.6.1) we have a canonical quasiisomorphism

$$\mathrm{gr}_n R\Gamma_{DR}(\mathcal{R}(X), M) \xrightarrow{\sim} R\Gamma_{DR}(\mathcal{R}(X)^o_n, M^{-n})[n]. \tag{3.11.7.2}$$

In particular, since $\dim \mathcal{R}(X)^o_n = n$, the cohomology groups $H^i_{DR}(\mathcal{R}(X), M)$[234] vanish for $i > 0$.

3.11.8 Lemma. The morphisms $H^a_{DR}(\mathcal{R}(X)_n, M) \to H^a_{DR}(\mathcal{R}(X), M)$ are surjective for $a > -n$ and are isomorphisms for $a > -n+1$. If X is affine then this is true, respectively, for $a > -n-1$ and $a > -n$.

Proof. We use (3.11.7.2). If X is affine then such are the varieties $\mathcal{R}(X)^o_m$. So the groups $H^a \mathrm{gr}_m R\Gamma_{DR}(\mathcal{R}(X), M)$ vanish for $a > -m$, which implies our statement. If X is compact, then we consider a coordinate projection $U^{(m)} \to X$. It is affine and X is a curve, so $H^a_{DR}(U^{(m)}, M^{-m}) = 0$ for $a > 1$. Since $H^a_{DR}(\mathcal{R}(X)^o_m, M^{-m}) \subset H^a_{DR}(U^{(m)}, M^{-m})$ we see that $H^a \mathrm{gr}_m R\Gamma_{DR}(\mathcal{R}(X), M) = 0$ for $a > -m+1$, and we are done. □

3.11.9. According to 3.5.11, a DG chiral (super) algebra A on X yields a (super) $\mathcal{D}$-complex $C(A)$ on $X^{\mathcal{S}}$ (the morphisms $\theta^{(\pi)}$ are defined by (3.5.11.3)). This complex is obviously admissible, i.e., $C(A)$ is a $\mathcal{D}$-complex on $\mathcal{R}(X)$. If A is a plain chiral algebra then $C(A)$ is a Cousin complex.

For the rest of §11 we assume that X is compact and connected.

Definition. We set $R\Gamma^{ch}(X, A) := R\Gamma_{DR}(\mathcal{R}(X), C(A))$ and $H^{ch}_a(X, A) = H^{ch}_a(A) := H^{-a} R\Gamma(X, A)$; this is the *chiral homology of X with coefficients in A* or, simply, the chiral homology of A.

We also set $\Gamma^{ch}(X, A) := \Gamma_{DR}(\mathcal{R}(X), A)$. According to 3.12.6 for a Dolbeaut algebra $\mathcal{Q}_X$ one has $R\Gamma^{ch}(X, A) = \Gamma^{ch}(X, \mathcal{Q}_X \otimes A)$.

[234] As well as $H^i_{DR}(\mathcal{R}(X)^o_n, M)$ and $H^i_{DR}(\mathcal{R}(X)_n, M)$.

One has $C(A)_{\mathcal{R}(X)^o_n} = \mathrm{Sym}^n(A[1])_{\mathcal{R}(X)^o_n}$, so the spectral sequence (3.11.6.2) converging to $H^{ch}_n(X,A)$ looks as

$$E^1_{p,q} = H^{-p-q}_{DR}(\mathcal{R}(X)^o_p, \mathrm{Sym}^p(A[1])). \tag{3.11.9.1}$$

The following notation is useful. For a right $\mathcal{D}$-module M on a connected smooth n-dimensional variety Y we set $H^{DR}_a(Y,M) := H^{n-a}_{DR}(Y,M)$; this is the *de Rham homology* of Y with coefficients in M. Notice that $H^{DR}_a(Y,M)$ vanish unless $a \in [-2n, 0]$; if Y is affine the vanishing holds unless $a \in [-2n, n]$. The vector space $H^{DR}_0(Y,M)$ vanishes if Y is non-compact; if Y is compact then there is a canonical morphism $M \to H^{DR}_0(X,M) \otimes \omega_Y$ which identifies the latter $\mathcal{D}$-module with the maximal "constant" quotient of M.[235]

So for a plain chiral algebra A we may rewrite our spectral sequence as

$$E^1_{p,q} = H^{DR}_{p+q}(\mathcal{R}(X)^o_p, \Lambda^p_{ext}A). \tag{3.11.9.2}$$

Here the $\mathcal{D}$-module $\Lambda^p_{ext}A$ on $\mathcal{R}(X)^o_p$ is $\mathrm{Sym}^p(A[1])_{\mathcal{R}(X)^o_p}[-p]$. Notice that $H^{ch}_a(X,A) = 0$ for $a < 0$.

Remarks. (i) The definition of $C(A)$ makes no use of the unit in A, so A can actually be any Lie^{ch} algebra.

(ii) We do not know if the chiral homology equals the left derived functor of H^{ch}_0. This is true if we deal with commutative chiral algebras (see 3.11.?).

3.11.10. For a plain chiral algebra A we will also denote the 0th chiral homology group $H^{ch}_0(A)$ by $\langle A\rangle = \langle A\rangle(X)$.[236] By construction and 3.5.12, 3.5.13 this is a plain $\mathcal{S}^\circ$ direct limit of vector spaces $H^{DR}_0(X^I, A_{X^I})$. So for every $I \in \mathcal{S}$ one has a canonical morphism of $\mathcal{D}$-modules

$$\langle\,\rangle_I : A^\ell_{X^I} \to \langle A\rangle \otimes \mathcal{O}_{X^I}. \tag{3.11.10.1}$$

These morphisms are compatible with pull-backs to the diagonals, so they form a morphism $\langle\,\rangle : A^\ell_{\mathcal{R}(X)} \to \langle A\rangle \otimes \mathcal{O}_{\mathcal{R}(X)}$ of the left $\mathcal{D}$-modules on $\mathcal{R}(X)$ (see 3.5.2 for terminology).

[235] See the proof of 2.4.2.

[236] According to 2.4.4 this notation is compatible with that of 2.4.1 when A is commutative.

Set $1_A^{ch} = 1^{ch} = \langle 1 \rangle \in \langle A \rangle$ (here 1 is the unit (horizontal) section of $A^{\ell}_{X^I}$).

Restricting $\langle\,\rangle$ to the complement $\mathcal{R}(X)^o_n$ of the diagonal divisor on $\mathrm{Sym}^n X$ we get the *correlator maps*

$$\langle\,\rangle_n : (\mathrm{Sym}^n A^{\ell})_{\mathcal{R}(X)^o_n} \to \langle A \rangle \otimes \mathcal{O}_{\mathcal{R}(X)^o_n}. \tag{3.11.10.2}$$

Remarks. (i) It follows easily from 3.11.8 that $H_0^{ch}(A) = \mathrm{Coker}(H_0^{DR}(U, j^*A^{\boxtimes 2}) \to H_0^{DR}(X, A))$ where the arrow comes from the chiral product $\mu : j_*j^*A \boxtimes A \to \Delta_* A$.

(ii) Assume that A is commutative. Then, by (i) and 2.4.4, $\langle A \rangle$ coincides with the same noted vector space from 2.4.1.

Recall that $\langle A \rangle$ is a commutative unital algebra (see 2.4.1). Its unit equals 1^{ch}, and the product $\cdot$ can be described (see, e.g., the proof of 2.4.4) in present terms as the quotient map of the composition $H_0^{DR}(X, A)^{\otimes 2} \xrightarrow{\boxtimes} H_0^{DR}(X \times X, A^{\boxtimes 2}) \to H_0^{DR}(X \times X, A_{X^2}) \to \langle A \rangle$ (the last arrow is the canonical morphism, the middle one comes since $A^{\boxtimes 2} \subset A_{X^2}$).

Example. For the unit algebra ω_X the obvious identification $\langle \omega_X \rangle = k$ (see 2.4.1) comes from the trace isomorphisms $H_0^{DR}(X^I, \omega_{X^I}) \xrightarrow{\sim} k$.

3.11.11. A filtration $A_0 \subset A_1 \subset ..$ on a chiral DG algebra A (see 3.3.11) yields an admissible filtration $C(A)_0 \subset C(A)_1 \subset ..$ on $C(A)$. Namely, for $I \in \mathcal{S}$ set $(A[1])^{\boxtimes I}_n := \sum \bigotimes_{i \in I} (A_{l_i}[1]) \subset (A[1])^{\boxtimes I}$, the summation is over the set of all collections $(l_i) \in \mathbb{Z}^I_{\geq 0}$ such that $\sum_{i \in I} l_i \leq n$. Now one has $C(A)_{nX^I} := \bigoplus_{T \in Q(I)} \Delta_*^{(I/T)} j_*^{(T)} j^{(T)*} (A[1])^{\boxtimes T}_n$ (see (3.5.11.1)).

Our filtration on $C(A)$ yields the filtrations on $\Gamma^{ch}(X, A)$ and $R\Gamma^{ch}(X, A)$. Since $\mathrm{gr}C(A) = C(\mathrm{gr}A)$ one has $\mathrm{gr}R\Gamma^{ch}(X, A) = R\Gamma^{ch}(X, \mathrm{gr}A)$. So one has a spectral sequence converging to $H^{ch}_{\cdot}(X, A)$ with

$$E^1_{p,q} = H^{ch}_{p+q}(X, \mathrm{gr}A)^p. \tag{3.11.11.1}$$

Here the upper index p is the grading on $H^{ch}(X, \mathrm{gr}A)$ that comes from the grading $\mathrm{gr}.A$.

Remark. The above construction makes sense if A is any DG Lie^{ch} algebra (see Remark (i) in 3.11.9) and the filtration need not be unital. For example, on any A we have a stupid canonical non-unital filtration $A_0 = 0$, $A_1 = A$. Then $C(A)_n \subset$

$C(A)$ is the submodule of sections supported on diagonals of dimension $\leq n$. It is clear that $\Gamma_{DR}(\mathcal{R}(X), C(A))_n \subset \Gamma_{DR}(\mathcal{R}(X), C(A)_n)$, and 3.11.4 (or its proof) shows that $R\Gamma_{DR}(\mathcal{R}(X), C(A))_n \to R\Gamma_{DR}(\mathcal{R}(X), C(A)_n)$ is a quasiisomorphism. Thus our filtration on $R\Gamma^{ch}(X, A)$ coincides with the canonical filtration and (3.11.11.1) equals (3.11.9.1).

Notice that this filtration is commutative (see 3.3.11).

We refer the reader to 3.12.7 for a review of needed facts on Batalin-Vilkovisky algebras.

3.11.12 Proposition. (i) For a DG chiral algebra[237] A the complex $\Gamma^{ch}(X, A)$ carries a canonical structure of a homotopy BV algebra.

(ii) For any *commutative* filtration on A the corresponding filtration on $\Gamma^{ch}(X, A)$ is compatible with the homotopy BV algebra structure.

Proof. (i) For $I \in \mathcal{S}$ set $\Gamma_{DR}C(A)_I := \Gamma_{DR}(X^I, C(A))$, so $\Gamma_{DR}C(A)$ is a functor on $\mathcal{S}^\circ$ with values in the DG category $C(k)$ of (super) k-complexes.

The category $C(k)^{\mathcal{S}}$ of all such functors is a DG (super) k-category. It carries a canonical tensor structure: for $F_i \in C(k)^{\mathcal{S}}$, $i \in I$, one has $(\otimes F_i)_J := \underset{J \twoheadrightarrow I}{\oplus} \otimes F_{iJ_i}$, the functor structure is the obvious one.

Notice that the pseudo-tensor structure on $C(k)^{\mathcal{S}}$ equals the one defined in 3.12.3(ii) (for $\mathcal{P} = \mathcal{S}^\circ$, $\mathcal{A} = C(k)$; the tensor operation Σ on $\mathcal{S}^\circ$ is the disjoint sum).

(a) Let us show that $\Gamma_{DR}C(A) \in C(k)^{\mathcal{S}}$ carries a canonical structure of BV algebra.

Namely, the product morphism $\Gamma_{DR}C(A) \otimes \Gamma_{DR}C(A) \stackrel{\cdot}{\to} \Gamma_{DR}C(A)$ at $J \in \mathcal{S}$ is the map $\underset{J=J_1 \sqcup J_2}{\oplus} \Gamma_{DR}(X^{J_1}, C(A)) \otimes \Gamma_{DR}(X^{J_2}, C(A)) \to \Gamma_{DR}(X^J, C(A))$ coming from the exterior tensor product maps $C(A)_{J_1} \boxtimes C(A)_{J_2} \to C(A)_J$. It is obviously commutative and associative. Let us show that $\cdot$ defines a BV algebra structure.

Let $\Gamma_{DR}C(A)[-1] \otimes \Gamma_{DR}C(A)[-1] \xrightarrow{\{\,\}} \Gamma_{DR}C(A)[-1]$ be the differential of $\cdot$. According to 3.5.11 it can be described as follows. For $J = J_1 \sqcup J_2 \in \mathcal{S}$ the component $\{\,\}_{J_1 J_2} : \Gamma_{DR}(X^{J_1}, C(A))[-1] \otimes \Gamma_{DR}(X^{J_2}, C(A))[-1] = \Gamma_{DR}(X^J, C(A)_{X^{J_1}}[-1] \boxtimes C(A)_{X^{J_2}}[-1]) \to \Gamma_{DR}(X^J, C(A))[-1]$ comes from a morphism $[\quad]_{J_1 J_2}$ of $\mathcal{D}$-complexes $C(A)_{X^{J_1}}[-1] \boxtimes C(A)_{X^{J_2}}[-1] \to C(A)_{X^J}[-1]$. For $T_i \in Q(J_i), T \in Q(J)$ the corresponding component

[237] The proposition remains valid in the non-unital setting of Remark in 3.11.11.

of this morphism $\Delta_*^{(J_1/T_1)} j_*^{(T_1)} j^{(T_1)*}(A[1])^{\boxtimes T_1}[-1] \boxtimes \Delta_*^{(J_2/T_2)} j_*^{(T_2)} j^{(T_2)*}(A[1])^{\boxtimes T_2}[-1] \to \Delta_*^{(J/T)} j_*^{(T)} j^{(T)*}(A[1])^{\boxtimes T}[-1]$ can be non-zero only if T is the quotient of $T_1 \sqcup T_2$ modulo a relation $t_1 = t_2$ where $t_1 \in T_1$, $t_2 \in T_2$. In this case our morphism is the $*$ bracket $A_{t_1} \boxtimes A_{t_2} \to \Delta_* A$ tensored by the identity map for $(A[1])^{\boxtimes T'}$ where $T' := T \setminus \{t\} = T_1 \setminus \{t_1\} \sqcup T_2 \setminus \{t_2\}$.

It is clear that $\{\,\}$ is an odd Poisson bracket with respect to $\cdot$, so $\Gamma_{DR} C(A)$ is a BV algebra in $C(k)^{\mathcal{S}}$ as promised.

(b) Let $\mathcal{T} = \mathcal{N}orm\, k[T]$ be the DG operad defined in 3.12.3(ii) for the tensor category $\mathcal{P} = \mathcal{S}^\circ$ (so T_I is the simplex whose vertices is the set of all "disjoint sum" functors $\Sigma_I : \mathcal{S}^I \to \mathcal{S}$). The projection $T \to$ (point) yields a canonical projection $\mathcal{T} \to \mathcal{C}om$ which is a quasiisomorphism.

One has a homotopy BV operad BV$\otimes\mathcal{T}$. According to (3.12.3.5) our BV algebra structure on $\Gamma_{DR} C(A)$ yields a BV$\otimes\mathcal{T}$ algebra structure on the complex $C(\mathcal{S}^\circ, \Gamma_{DR} C(A)) = \Gamma^{ch}(X, A)$. We are done.

(ii) Clear from the construction. □

3.11.13. (i)[238] Let A be a DG chiral algebra equipped with a commutative filtration $A_\cdot$. We get a filtered complex $R\Gamma^{ch}(X, A)$. Replacing $A_\cdot$ by $\Omega_X \otimes A_\cdot$ as in 3.12.6 we get on $R\Gamma^{ch}(X, A)$ a canonical structure of homotopy BV algebra. It is clear that $R\Gamma^{ch}$ sends filtered quasiisomorphisms to filtered quasiisomorphisms.

Denote by $\mathcal{FCA}(X)$ the category of DG chiral algebras equipped with a commutative filtration; let ${}^c\mathcal{FCA}(X)$ be the full subcategory of algebras with commutative filtrations. Below we denote by $\mathcal{H}o$ the localization of our categories by (filtered) quasiisomorphisms.[239] We see that $R\Gamma^{ch}$ lifts canonically to a functor

$$R\Gamma^{ch} : \mathcal{H}o\, {}^c\mathcal{FCA}(X) \to \mathcal{H}o\mathcal{BV}. \tag{3.11.13.1}$$

(ii) If A is a commutative DG algebra then the bracket $\{\,\}$ on its chiral complex vanishes, so $R\Gamma^{ch}(X, A)$ is a homotopy commutative algebra. So $H^{ch}_\cdot(A)$ is a commutative (super)

[238] This subsection makes sense in the non-unital setting.

[239] We did not check that our categories are closed model categories, so this definition is a bit formal.

algebra. We denote its product by $\cdot$. According to Remark (ii) from 3.11.10 on $H_0^{ch}(A)$ this product coincides with the product on $\langle A\rangle$ from 2.4.1. Thus $H_0^{ch}(A)$ is a unital algebra with unit 1^{ch}. We will see in 3.11.16 that 1^{ch} is, in fact, the unit for the whole $H_\cdot^{ch}(A)$.

3.11.14. Let A, B be DG chiral algebras. Choose a "disjoint union" functor $\sqcup \in T_2$ (see section (b) of the proof of 3.11.12 and 3.12.3(ii));[240] it yields a product $\cdot = \cdot_\sqcup$ on $\Gamma^{ch}(X, A\otimes B)$. This product need not commute with the differential. However, since the canonical morphisms $A, B \to A\otimes B$ mutually commute, their composition with $\cdot$

$$\circ_{A,B} : \Gamma^{ch}(X,A)\otimes\Gamma^{ch}(X,B)\to\Gamma^{ch}(X,A\otimes B) \tag{3.11.14.1}$$

is a morphism of complexes.

Passing as above to derived functors (i.e., replacing A by $\Omega_X\otimes A$, etc.) we can replace Γ^{ch} by $R\Gamma^{ch}$. The morphism $\circ_{A,B}$ for $R\Gamma^{ch}$ does not depend on the choice of $\sqcup$.

Remarks. (i) Assume that A, B carry filtrations and let us equip $A\otimes B$ with the tensor product of this filtrations. Then $\circ_{A,B}$ is a morphism of filtered complexes.

(ii) To make (3.11.14.1) independent of the choice of $\sqcup$ one writes it as a canonical morphism $\Gamma^{ch}(X,A)\otimes\Gamma^{ch}(X,B)\otimes\mathcal{T}_2\to\Gamma^{ch}(X,A\otimes B)$ (for $\mathcal{T}_2$ see the proof of 3.11.12(i)(b)).

(iii) $\circ_{A,B}$ is a morphism of homotopy BV algebras up to canonical homotopies. Precisely, for a finite non-empty I let the simplicial set T'_I be the joint of $T_I\times T_I$ and T_I (the simplex T_I was defined in the proof of 3.11.12(i)(b)). This is a contractible simplicial set equipped with embeddings $\nu_0 : T_I\times T_I\hookrightarrow T'_I$, $\nu_1 : T_I\hookrightarrow T'_I$. Set $\mathcal{T}'_I := \mathcal{N}ormk[T'_I]$. Now there is a canonical morphism $\alpha : \mathrm{BV}_I\otimes\mathcal{T}'_I\otimes\Gamma^{ch}(X,A)\otimes\Gamma^{ch}(X,B)\to\Gamma^{ch}(X,A\otimes B)$ such that $\alpha\nu_0 = \circ_{A,B}a_0$ and $\alpha\nu_1 = a_1\circ_{A,B}$. Here a_0 is the action of $\mathrm{BV}\otimes\mathcal{T}\otimes\mathcal{T}$ on $\Gamma^{ch}(X,A)\otimes\Gamma^{ch}(X,B)$ (see 3.12.6), and a_1 is the action of $\mathrm{BV}\otimes\mathcal{T}$ on $\Gamma^{ch}(X,A\otimes B)$. In the filtered setting the above canonical homotopies are compatible with filtrations. The compatibility with compositions of operations is left to the reader.

3.11.15 Theorem. (i) Assume that A, B are $\mathcal{O}_X$-flat.[241] Then the product morphism $\circ_{A,B} : R\Gamma^{ch}(X,A)\otimes R\Gamma^{ch}(X,B)\to R\Gamma^{ch}(X,A\otimes B)$ is a quasiisomorphism. If A, B

[240] Recall that $\sqcup$ is defined up to a canonical isomorphism, and we need to choose a concrete functor.

[241] It suffices to assume that A and B are Tor-independent, i.e., the supports of their $\mathcal{O}_X$-torsion are disjoint.

are equipped with filtrations[242] such that $\mathrm{gr}A$, $\mathrm{gr}B$ are $\mathcal{O}_X$-flat then this is a filtered quasiisomorphism.

(ii) One has $H^{ch}_{\neq 0}(X,\omega)=0$, so (see Example in 3.11.10) one has a canonical identification $R\Gamma^{ch}(X,\omega)=H^{ch}_0(\omega)=k$.

(iii) The product endomorphism $a\mapsto a\circ_{A,\omega} 1^{ch}_{\omega}$ of $R\Gamma^{ch}(X,A)$ is identity.

Remark. The vanishing statement (ii) follows from the contractability of $\mathcal{R}(X)$ in classical topology (see 3.5.1(iv)) combined with 3.11.1(iv) and usual comparison between the de Rham and topological homology. The argument below is an algebraic version of the topological proof of 3.5.1(iv).

Proof. Since $\mathrm{gr}R\Gamma^{ch}(X,A)=R\Gamma^{ch}(X,\mathrm{gr}A)$ (see 3.11.11) and $\mathrm{gr}(A\otimes B)=\mathrm{gr}A\otimes\mathrm{gr}B$ the filtered setting follows from the non-filtered one. So we forget about filtrations.

Let us construct the inverse to $\circ_{A,B}$ explicitely.

Consider the $\mathcal{S}^\circ\times\mathcal{S}^\circ$-diagram of varieties $X^{\mathcal{S}}\times X^{\mathcal{S}}$. Consider $\mathcal{D}$-complexes $C(A,B)$ and $C(A)\boxtimes C(B)$ on $X^{\mathcal{S}}\times X^{\mathcal{S}}$ where $C(A,B)_{X^I\times X^J}$ is the Cousin complex of $A_{X^I}\boxtimes B_{X^J}[|I\sqcup J|]$ with respect to the diagonal stratification of $X^{I\sqcup J}$ and $(C(A)\boxtimes C(B))_{X^I\times X^J}:=C(A)_{X^I}\boxtimes C(B)_{X^J}$. The flatness property of A, B assures that the obvious embedding $C(A)\boxtimes C(B)\hookrightarrow C(A,B)$ is a quasiisomorphism. The cohomology of $C(A)\boxtimes C(B)$ are computed by Kunneth formula, so one has a canonical identification $R\Gamma_{DR}(X^{\mathcal{S}}\times X^{\mathcal{S}},C(A,B))=R\Gamma^{ch}(X,A)\otimes R\Gamma^{ch}(X,B)$.

Consider the diagonal and disjoint union functors $\mathcal{S}\xrightarrow{\delta}\mathcal{S}\times\mathcal{S}\xrightarrow{\sqcup}\mathcal{S}$, and their liftings to the morphisms of the diagrams $X^I\xrightarrow{\delta}X^I\times X^I$, $X^I\times X^J\xrightarrow{\sim}X^{I\sqcup J}$ and of the $\mathcal{D}$-complexes $\delta_*C(A\otimes B)_{X^I}\xrightarrow{\delta_{A,B}}C(A,B)_{X^I\times X^I}$, $C(A,B)_{X^I\times X^J}\xrightarrow{\circ_{A,B}}C(A\otimes B)_{X^{I\sqcup J}}$. Here $\delta_{A,B}$ comes from the canonical identification $\delta^!C(A,B)_{X^I\times X^I}=C(A\otimes B)$ and $\circ_{A,B}$ from the morphism $1_B^{\boxtimes I}\boxtimes 1_A^{\boxtimes J}: A_{X^I}\boxtimes B_{X^J}\to(A\otimes B)_{X^{I\sqcup J}}$. Passing to $R\Gamma_{DR}$'s we get the morphisms of complexes

$$R\Gamma^{ch}(X,A\otimes B)\underset{\circ_{A,B}}{\overset{\delta_{A,B}}{\rightleftarrows}}R\Gamma^{ch}(X,A)\otimes R\Gamma^{ch}(X,B). \tag{3.11.15.1}$$

It is clear that this $\circ_{A,B}$ coincides with the morphism (3.11.14.1).

[242] The filtrations are assumed to be unital.

We claim that $\delta_{A,B}$ and $\circ_{A,B}$ are mutual inverses. By the definition of Γ the composition $\circ_{A,B}\delta_{A,B}$ is the identity endomorphism of $R\Gamma^{ch}(X, A\otimes B)$. To finish we need first to prove statements (ii) and (iii) of the theorem:

(ii) Consider the case $A = B = \omega_X$. We are playing with a commutative chiral algebra, so $\circ_{\omega,\omega}$ is the commutative and associative product $\cdot$ on $H^{ch}_\cdot(\omega)$ from 3.11.13(ii).

To prove the vanishing take a to be the minimal number > 0 such that $H^{ch}_a(\omega) \neq 0$. Since $H^{ch}_0(\omega) = k$ we can rewrite, by Kunneth, the H_a maps of (3.11.15.1) as

$$H^{ch}_a(\omega) \xrightarrow{\delta_a} H^{ch}_a(\omega) \oplus H^{ch}_a(\omega) \xrightarrow{\cdot_a} H^{ch}_a(\omega).$$

Here δ_a is the diagonal embedding (since its composition with the projections to the multiples is identity). Both components of $\cdot_a$ are equal to the endomorphism $\gamma \mapsto 1^{ch}\cdot\gamma$ of $H^{ch}_a(\omega)$. We will prove that it is the identity map. Then $\cdot_a\delta_a$ is multiplication by 2, which contradicts to the fact that $\cdot_a\delta_a$ is the identity map.

Indeed, we know that 1^{ch} is the unit in $H^{ch}_0(\omega_X)$. Therefore one has $1^{ch}\cdot(1^{ch}\cdot\gamma) = (1^{ch}\cdot 1^{ch})\cdot\gamma = 1^{ch}\cdot\gamma$. So, since also $\gamma = \cdot_a\delta_a(\gamma) = 2(1^{ch}\cdot\gamma)$, we see that $1^{ch}\cdot\gamma = \gamma$, q.e.d.

(iii) Assume that A is arbitrary, $B = \omega$. Recall that the identification $\mathrm{tr} : R\Gamma^{ch}(X,\omega) \xrightarrow{\sim} k$ of (ii) is given by the trace maps $R\Gamma_{DR}(X^I, C(\omega)_{X^I}) = R\Gamma_{DR}(X^I, \omega_{X^I})[I] \to k$. Therefore the composition $R\Gamma^{ch}(X,A) \xrightarrow{\delta_{A,\omega}} R\Gamma^{ch}(X,A)\otimes R\Gamma^{ch}(X,\omega) \xrightarrow{id\otimes\mathrm{tr}} R\Gamma^{ch}(X,A)$ is the identity morphism. Thus $\delta_{A,\omega}$ is an isomorphism. Since $\circ_{A,\omega}$ is left inverse to $\delta_{A,\omega}$, we see that $\delta_{A,\omega}$, $\circ_{A,\omega}$ are mutually inverse and $a \mapsto a\circ_{A,\omega} 1^{ch}$ is the identity endomorphism of $R\Gamma^{ch}(X,A)$.

Now we are ready to prove that $\circ_{A,B}$ and $\delta_{A,B}$ are mutually inverse for arbitrary DG chiral algebras A, B. The composition $\delta_{A,B}\circ_{A,B}$ comes from the composition of morphisms $\delta_* C(A)_{X^I}\boxtimes\delta_* C(B)_{X^J} \to C(A,\omega)_{X^I\times X^I}\boxtimes C(\omega,B)_{X^J\times X^J} \leftarrow [C(A)_{X^I}\boxtimes C(\omega)_{X^I}]\boxtimes [C(\omega)_{X^J}\boxtimes C(B)_{X^J}] = [C(A)_{X^I}\boxtimes C(\omega)_{X^J}]\boxtimes[C(\omega)_{X^I}\boxtimes C(B)_{X^J}] \to C(A)_{X^{I\sqcup J}}\boxtimes C(B)_{X^{I\sqcup J}}$. Here the first arrow is $\delta_{A,\omega}\boxtimes\delta_{\omega,B}$, the second (left) arrow is the tensor product of the quasiisomorphisms from (i), the last arrow is $\circ_{A,\omega}\boxtimes\circ_{\omega,B}$. These morphisms of $\mathcal{D}$-complexes lift the obvious morphisms of diagrams $X^S\times X^S \to X^S\times X^S\times X^S\times X^S \to X^S\times X^S\times X^S\times X^S \to X^S\times X^S$. Now apply $R\Gamma_{DR}$; we are done by (iii). $\square$

3.11.16 Remarks. (i) The definition of $\circ_{A,B}$ from 3.11.14 generalizes immediately to any number of DG chiral algebras. It is obviously homotopy associative and commutative. So for any finite collection of DG chiral algebras $\mathcal{A}_\alpha$ one has a canonical quasiisomorphism

$$\circ = \circ_{\{A_\alpha\}} : \otimes R\Gamma^{ch}(X, A_\alpha) \xrightarrow{\sim} R\Gamma^{ch}(X, \otimes A_\alpha). \tag{3.11.16.1}$$

If A_α are equipped with filtrations and $\otimes A_\alpha$ with the tensor product of these filtrations then (3.11.16.1) is a filtered quasiisomorphism. So we have tensor functors

$$R\Gamma^{ch} : \mathcal{H}o\mathcal{CA}(X) \to D(k), \quad R\Gamma^{ch} : \mathcal{H}o\mathcal{FCA}(X) \to DF(k). \tag{3.11.16.1}$$

(ii) According to Remark (ii) in 3.11.14 and 3.12.? we have a tensor functor

$$R\Gamma^{ch} : \mathcal{H}o^c\mathcal{FCA}(X) \to \mathcal{H}o\mathcal{BV}. \tag{3.11.16.2}$$

(iii)

(iv) One may identify $H_0^{ch}(X, \omega)$ with the 0th de Rham cohomology supported at any point of $\mathcal{R}(X)$. This gives the following description of the map $\circ_{\omega,A}$. Take any finite non-empty $T \subset X$, so we have $[T] \in \mathcal{R}(X)$ and the corresponding point $[T] \in X^T$. Let 1_T be the generator of $H^0_{DR\,[T]}(X^T, C(\omega)) = k$ (cohomology with support at $[T]$). The morphisms $1_A^{\boxtimes T} : C(\omega)_{X^T} \boxtimes C(A)_{X^I} \to C(A)_{X^{T\sqcup I}}$ yield the morphisms of complexes $R\Gamma(X^{\mathrm{S}}, C(A)) \to R\Gamma_{DR\,[T]}(X^T, C(\omega)) \boxtimes R\Gamma(X^{\mathrm{S}}, C(A)) \xrightarrow{\sim} R\Gamma_{DR}(X^T \times X^{\mathrm{S}}, C(\omega)_{X^T} \boxtimes C(A)) \to R\Gamma_{DR}(X^{\mathrm{S}}, C(A))$ where the first arrow is $1_T \boxtimes \cdot$. Our $\circ_{\omega,A}$ is the composition of these maps.

Consider the case of 0th chiral homology. Let $S, T \subset X$ be disjoint finite subsets. Then, by 3.11.15(iii), for every $v \in \otimes A_s^\ell$ one has[243]

$$\langle (\otimes 1_t) \otimes (\otimes v_s) \rangle_{T\sqcup S} = \langle \otimes v_s \rangle_S. \tag{3.11.16.3}$$

[243] See 3.11.10 for notation.

3.11.17 Lemma. For every affine open $j : U \hookrightarrow X$ and a DG chiral algebra A one has $H^{\cdot}_{DR}(\mathcal{R}(U), C(A)) = 0$.

Proof. The operator $\circ = \circ_{\omega, j_* j^* A}$ acts on $H^a_{DR}(\mathcal{R}(X), C(j_* j^* A)) = H^{ch}_{-a}(X, j_* j^* A)$ as identity by 3.11.15(iii). On the other hand, we can repeat the construction of $\circ$ from 3.11.14 replacing X by U. We get a map $\circ_U : H^0_{DR}(\mathcal{R}(U), C(\omega)) \otimes H^a_{DR}(\mathcal{R}(U), C(A)) \to H^a_{DR}(\mathcal{R}(U), C(A))$. It is clear that the restriction isomorphism $H^a_{DR}(\mathcal{R}(X), C(j_* j^* A)) \xrightarrow{\sim} H^a_{DR}(\mathcal{R}(U), C(A))$ identifies $\circ$ and $\circ_U$, i.e., for $v \in H^a_{DR}(\mathcal{R}(X), C(j_* j^* A))$ one has $\circ(v)|_U = \circ_U(1^{ch}|_U \otimes v|_U)$. Since U is affine one has $1^{ch} = 0$[244] hence $\circ = 0$. We are done. □

Remark. The above lemma has the following relative version. Let Y be a smooth variety and $j : U \hookrightarrow X \times Y$ an open embedding such that U is affine over Y. An obvious relative version of the above constructions provides a $\mathcal{D}$-complex $C(j_* j^* A)$ on $\mathcal{R}(X) \times Y$. We claim that $p_{Y*} C(j_* j^* A) \in D\mathcal{M}(Y)$ vanishes. The proof coincides with that of the lemma.

3.11.18 Theorem. For DG chiral algebras A, B the projections $A \times B \to A, B$ yield a canonical isomorphism $R\Gamma^{ch}(X, A \times B) \xrightarrow{\sim} R\Gamma^{ch}(X, A) \times R\Gamma^{ch}(X, B)$.

Proof. We have an obvious embedding $C(A) \oplus C(B) \hookrightarrow C(A \times B)$, so the $\mathcal{D}$-complex $C(A) \oplus C(B)$ is canonically a direct summand of $C(A \times B)$. Let F be its complement, i.e., the kernel of the projection $C(A \times B) \to C(A) \oplus C(B)$. It suffices to show that $H^{\cdot}_{DR}(\mathcal{R}(X), F)$ vanish.

One has $R\Gamma_{DR}(\mathcal{R}(X), F) = R\Gamma_{DR}(\mathcal{R}(X) \times \mathcal{R}(X), P)$ where P is the following $\mathcal{D}$-complex P on $\mathcal{R}(X) \times \mathcal{R}(X)$ (i.e., on $X^{\mathcal{S}} \times X^{\mathcal{S}}$). Let $j^{[I,J]} : U^{[I,J]} \hookrightarrow X^I \times X^J$ be the complement to the diagonals $x_i = x_j$, $i \in I$, $j \in J$. Then $P_{X^I \times X^J} := j^{[I,J]}_* j^{[I,J]*} C(A, B)_{X^I \times X^J}$ where the complex $C(A, B)$ was defined in the proof of 3.11.15. We compute the cohomology by means of our usual spectral sequence with respect to the second variable. Its terms are $H^a_{DR}(\mathcal{R}(X) \times \mathcal{R}(X)^o_n, P)$. Let $\tilde{U}_n \subset \mathcal{R}(X) \times \mathcal{R}(X)^o_n$ be the open subset of pairs $([S], [T])$ such that $S \cap T = \emptyset$; then $H^a_{DR}(\mathcal{R}(X) \times \mathcal{R}(X)^o_n, P) = H^a_{DR}(\tilde{U}_n, C(A) \boxtimes \mathrm{Sym}^n B)$. Let $p_1, p_2 : \tilde{U}_n \to \mathcal{R}(X), \mathcal{R}(X)^o_n$ be the projections. We may rewrite our group as $H^a_{DR}(\mathcal{R}(X)^o_n, (Rp_{2*} p_1^* C(A)) \overset{L}{\otimes} \mathrm{Sym}^n B)$. According to the remark after 3.11.17 one has

[244] Indeed, $H^0_{DR}(\mathcal{R}(U), C(\omega)) = 0$ say, by 3.11.8.

$Rp_{2*}p_1^*C(A) = 0$, and we are done. □

3.11.19. Let us formulate the "localization principle" for chiral homology.

Let $S \subset X$ be a finite non-empty subset. Consider the subspace $i_S : \mathcal{R}(X,S) \hookrightarrow \mathcal{R}(X)$ of those points that the corresponding finite subsets of X contain S.

Let $i_S : X^{\mathrm{S}}_S \hookrightarrow X^{\mathrm{S}}$ be the r preimage of $\mathcal{R}(X,S)$; this is a diagram of closed subvarieties of X^{S}. The !-sheaves, right $\mathcal{D}$-modules, etc., on X^{S}_S are the same as corresponding objects on X^{S} supported on X^{S}_S. We have the standard functors $Ri^!_S$, i_{S*} between the corresponding derived categories.

Let A be a DG chiral algebra. Consider the canonical arrow

$$R\Gamma_{DR}(\mathcal{R}(X,S), Ri^!_S C(A)) \to R\Gamma_{DR}(\mathcal{R}(X), C(A)). \tag{3.11.19.1}$$

3.11.20 Proposition. This is a quasiisomorphism.

Proof. Idea of the argument: consider the complementary open subset $\mathcal{R}(X) \smallsetminus \mathcal{R}(X,S) \subset \mathcal{R}(X)$. By the standard exact sequence our claim amounts to the vanishing of the cohomology $H^{\cdot}_{DR}(\mathcal{R}(X) \smallsetminus \mathcal{R}(X,S), C(A))$.

Notice that $\mathcal{R}(X) \smallsetminus \mathcal{R}(X,S)$ is covered by open subsets $\mathcal{R}(U_s)$, $s \in S$, where $U_s := X \smallsetminus \{s\}$. Any intersection of these subsets is $U_T := X \smallsetminus T$ for some non-empty $T \subset S$. All cohomology groups $H^{\cdot}_{DR}(\mathcal{R}(U_T), C(A))$ vanish by 3.11.15, so, by the Cech spectral sequence, the cohomology $H^{\cdot}_{DR}(\mathcal{R}(X) \smallsetminus \mathcal{R}(X,S), C(A))$ also vanish.

To make the above argument precise one has to play with $H^{\cdot}_{DR}(\mathcal{R}(X) \smallsetminus \mathcal{R}(X,S), C(A))$ using the sheaves on $X^{\mathrm{S}} \smallsetminus X^{\mathrm{S}}_S$. The details are left to the reader. □

Remark. As follows from 3.11.16(iv) the inverse map to (3.11.19.1) equals the composition of arrows (see 3.11.16(iv)) $R\Gamma(X^{\mathrm{S}}, C(A)) \to R\Gamma_{DR[S]}(X^S, C(\omega)) \boxtimes R\Gamma(X^{\mathrm{S}}, C(A)) \xrightarrow{\sim} R\Gamma_{DR[S]\times X^{\mathrm{S}}}(X^S \times X^{\mathrm{S}}, C(\omega)_{X^S} \boxtimes C(A)) \to R\Gamma_{DR\,X^{\mathrm{S}}_S}(X^{\mathrm{S}}, C(A))$.

3.11.21. Assume we are in situation 3.11.19. Set $U = U_S := X \smallsetminus S$. We have a retraction $\rho_S : \mathcal{E}xp(X) \to \mathcal{R}(X,S)$, $[T] \mapsto [S \cup T]$. It induces a bijective map[245] $\mathcal{E}xp(U) \to \mathcal{R}(X,S)$.

[245] We replaced $\mathcal{R}$ by $\mathcal{E}xp$ for this reason.

The restriction of ρ_S to each stratum $\mathcal{E}xp(U)^o_n$ is a closed embedding of smooth algebraic varieties $\rho_{Sn} : \mathcal{E}xp(U)^o_n \hookrightarrow \mathcal{R}(X)^o_{n+|S|}$; its image is $\mathcal{R}(X,S) \cap \mathcal{R}(X)^o_{n+|S|}$.

Assume that our A is flat at points of S. According to 3.11.20 the chiral cohomology of A are de Rham cohomology of the $\mathcal{D}$-complex $C(A,S) := i_{S*}Ri^!_S C(A)$. Look at the spectral sequence (3.11.6.2). The complex $C(A,S)_{\mathcal{R}(X)^o_{n+|S|}}$ is supported on the image of ρ_{Sn}. One has $i^!_{Sn}C(A,S) = (\otimes A^\ell_s) \otimes \mathrm{Sym}^n(A[1])_{\mathcal{E}xp(U)^o_n}$. Therefore our spectral sequence converging to $H^{ch}_n(X,A)$ is (we shifted the p index in (3.11.6.2) by $|S|$)

$$E^1_{p,q} = (\otimes A^\ell_s) \otimes H^{-p-q}_{DR}(\mathcal{E}xp(U)^o_p, \mathrm{Sym}^p(A[1])). \tag{3.11.21.1}$$

If A is a plain chiral algebra we may rewrite it as (cf. (3.11.9.2))

$$E^1_{p,q} = (\otimes A^\ell_s) \otimes H^{DR}_{p+q}(\mathcal{E}xp(U)^o_p, \Lambda^p_{ext}A). \tag{3.11.21.2}$$

Since $\mathcal{E}xp(U)^o_i$ are affine varieties of dimension i one has $E^1_{p,q} = 0$ unless $p \geq q \geq 0$.

Now we can describe the 0th chiral homology group $\langle A \rangle$ for a plain chiral algebra A. Consider the Lie algebra $h(U,A) = H^0_{DR}(U,A)$. It acts canonically on each vector space A^ℓ_s (recall that $i_{s*}A^\ell_s$ is a $j_{U*}j^*_U A$-module, see 3.4.3(i)), hence $\otimes A^\ell_s$ is an $h(U,A)$-module.

3.11.22 Proposition. (i) $\langle A \rangle$ is canonically isomorphic to the coinvariants of the $h(U,A)$-action on $\otimes A^\ell_s$.

(ii) Let $P \subset A_U$ be any $\mathcal{D}_U$-submodule which generates A_U as a chiral algebra. Then the coinvariants for the actions of $h(U,P)$ and $h(U,A)$ on $\otimes A^\ell_s$ coincide.[246] Thus

$$\langle A \rangle = (\otimes A^\ell_s)_{h(U,P)}. \tag{3.11.22.1}$$

Proof. (i) As follows from (3.11.21.2) one has $\langle A \rangle_X = E^2_{0,0} = \mathrm{Coker}(d : E^1_{1,0} \to E^1_{1,0})$. The latter differential is the arrow $h(U,A) \otimes (\otimes A^\ell_s) \to \otimes A^\ell_s$. One checks immediately that this is the action map, and we are done.

(ii) We want to show that the projection $\langle A \rangle_{h(U,P)} \twoheadrightarrow \langle A \rangle_{h(U,A)}$ is an isomorphism. Let F be the image of the action map $h(U,P) \otimes (\otimes A_s) \to \otimes A_s$. We use induction by $|S|$.

[246] i.e., the images of the action maps $h(U,P) \otimes (\otimes A_s) \to \otimes A_s$ and $h(U,A) \otimes (\otimes A_s) \to \otimes A_s$ coincide.

(a) Assume that $|S| = 1$, $S = \{s\}$. Set $Q := \mu(j_*j^*P \boxtimes P)$. It suffices to show that the image G of the action map $h(U,Q) \otimes A_s^\ell \to A_s^\ell$ is contained in F. The map $h(\mu) : h(U \times U, j_*j^*P \boxtimes P) \to h(U,Q)$ is surjective, so G equals the image of the composition $h(U \times U, j_*j^*P \boxtimes P) \otimes A_s \xrightarrow{\mu \otimes id_{A_s}} h(U,Q) \otimes A_s \to A_s$. By Jacobi this map equals the sum of two maps each of them is composition $h(U \times U, j_*j^*P \boxtimes P) \otimes A_s \to h(U,P) \otimes A_s \to A_s$ where the first arrow is the action along the first or the second variable and the second one is the action map. The images of these maps are contained in F, and we are done.

(b) Assume that $|S| > 1$. Pick $s_0 \in S$; set $S' = S \setminus \{s_0\}$, so we have $U \subset U' := X \setminus S'$. Let $P' \subset A_{U'}$ be a $\mathcal{D}_{U'}$-submodule which equals P on U. We can assume that P' generates $A_{U'}$. Indeed, let $A' \subset A$ be the chiral subalgebra that equals A on $X \setminus \{s_0\}$ and is the chiral subalgebra generated by P' on U'. Since $\underset{s \in S}{\otimes} A_s$ is a quotient of $\underset{s \in S}{\otimes} A'_s$ and we consider coinvariants of the same vector space of operators the desired statement for (A,S) follows from that for (A',S). We arrive to our assumption replacing A by A'.

By (i) we want to prove that the embedding $F \subset K := \mathrm{Ker}(\underset{s \in S}{\otimes} A_s \twoheadrightarrow \langle A \rangle)$ is an isomorphism. Since $h(U,P)$ generates $A_{s_0}^{a\,s}$ one has $\underset{s \in S}{\otimes} A_s = F + \underset{s \in S'}{\otimes} A_s$. Since $\underset{s \in S'}{\otimes} A_s \twoheadrightarrow \langle A \rangle$ it suffices to check that $F \cap (\underset{s \in S'}{\otimes} A_s) \hookrightarrow K \cap (\underset{s \in S'}{\otimes} A_s)$ is an isomorphism.

By induction assumption and (i) applied to S' we know that the image F' of $h(U',P') \otimes (\underset{s \in S'}{\otimes} A_s) \to \underset{s \in S'}{\otimes} A_s$ equals the kernel of the projection $\underset{s \in S'}{\otimes} A_s \twoheadrightarrow \langle A \rangle$, i.e., to $K \cap (\underset{s \in S'}{\otimes} A_s)$. Since $F \supset F'$ we are done. □

Remark. The above canonical projection $\otimes A_s^\ell \to \langle A \rangle$ coincides with the value at $[S] \in \mathcal{R}(X)^o_{|S|}$ of the correlator map $\langle\,\rangle_{|S|}$ (see (3.11.10.2)).

3.11.23. Let now A be a DG chiral algebra and $\{M_s\}$, $s \in S$, a finite family of DG chiral A-modules. We define *the chiral homology* $H_a^{ch}(A, \{M_s\})$ *of* A *with coefficients in* $\{M_s\}$ as follows.

Consider the chiral algebra $\tilde{A} := A \oplus (\oplus M_s[-1])$ (see 3.3.5). This chiral algebra carries an obvious $\mathbb{G}_m^S$-action, therefore $C(\tilde{A})$ is a $\mathbb{Z}^S$-graded complex. Denote by $C(A, \{M_s\})$ its component of degree 1^S. Set

$$R\Gamma^{ch}(X, A, \{M_s\}) := R\Gamma_{DR}(\mathcal{R}(X), C(A, \{M_s\})) \tag{3.11.23.1}$$

and $H^{ch}_a(X, A, \{M_s\}) = H^{ch}_a(A, \{M_s\}) := H^{-a}R\Gamma^{ch}(X, A, \{M_s\})$. If $S = \emptyset$ these are the chiral homology of A as defined in 3.11.9.

Let us describe $R\Gamma^{ch}(X, A, \{M_s\})$ a bit more explicitly. Denote by $\mathfrak{S}_S$ the category whose objects are non-empty finite sets equipped with an embedding $S \hookrightarrow I$, morphisms are surjections identical on S. We have a $\mathfrak{S}^\circ_S$-diagram of closed embeddings $X^{\mathfrak{S}_S}$, $I \mapsto X^I$ which carries a $\mathcal{D}$-complex $C'(A, \{M_s\})$ defined as follows. The graded $\mathcal{D}$-module $C'(A, \{M_s\})_{X^I}$ has an additional grading by $Q(I, S) :=$ the set of[247] morphisms $\pi : I \to T$ in $\mathfrak{S}_S$. For such π the corresponding component is $\Delta^{(\pi)}_* j^{(I)}_* j^{(I)*}((A[1])^{\boxtimes T \smallsetminus S} \boxtimes (\boxtimes M_s))[|T \smallsetminus S|]$. The differential is the sum of two components: the first one comes from μ_A and μ_{M_s}, the second one from the differentials on A and M_s. We have an obvious canonical identification $R\Gamma_{DR}(X^{\mathfrak{S}_S}, C'(A, \{M_s\}) = R\Gamma^{ch}(X, A, \{M_s\})$.

It is clear that $R\Gamma^{ch}(X, A, \{M_s\})$ is a homology functor of the arguments $\{M_s\}$. Moreover, chiral homology are functorial with respect to chiral operations: for $S \twoheadrightarrow T$ and a T-family of A-modules N_t every $\varphi \in P^{ch}_{A\,S/T}(\{M_s\}, \{N_t\}) := \otimes P^{ch}_{A\,S_t}(\{M_s\}, N_t)$ (see 3.3.4) yields an obvious morphism $R\Gamma^{ch}(\varphi) : R\Gamma^{ch}(X, A, \{M_s\}) \to R\Gamma^{ch}(X, A, \{N_t\})$; one has $R\Gamma^{ch}(\varphi\psi) = R\Gamma^{ch}(\varphi)R\Gamma^{ch}(\psi)$.

We leave it to the reader to generalize 3.11.21 to the present setting. Notice that the analog of 3.11.20 for $C(A, \{M_s\})$ is true since $C(A, \{M_s\})$ is a direct summand of $C(\tilde{A})$.

3.11.24. Consider the situation when $S \neq \emptyset$ and each M_s is supported at a (closed) point $s \in X$.[248] One may assume that these points are pairwise different (otherwise $C(A, \{M_s\}) = 0$), so we have $S \subset X$. Let $j_S : U = U_S = X \setminus S \hookrightarrow X$ be the complementary embedding. According to 3.4.3(i) the A-action on M_s extends canonically to a $j_{S*}j^*_S A$-action. The obvious morphism $C(A, \{M_s\}) \to C(j_{S*}j^*_S A, \{M_s\})$ is an isomorphism, so we can write

$$H^{ch}_a(U, A, \{M_s\}) := H^{ch}_a(X, A, \{M_s\}). \tag{3.11.24.1}$$

[247] isomorphisms classes of

[248] Notice that the general situation of 3.11.23 reduces to that one: for arbitrary $\{M_s\}$ the complex $R\Gamma^{ch}(A, \{M_s\})$ is equal to the de Rham cohomology of $U^{(S)}$ with coefficients in a $\mathcal{D}$-complex with fibers $R\Gamma^{ch}(A, \{M_{x_s}\})$.

Recall (see 3.4.5) that an A-module M_s supported at s amounts to a discrete $A_s^{a\,s}$-module $h(M_s) = i_s^!(M_s)$. In formulas we will often replace M_s by $h(M_s)$.

The complex $C(A, \{M_s\})$ is supported on $\mathcal{R}(X, S) \subset \mathcal{R}(X)$. As in 3.11.21, the spectral sequence (3.11.6.2) (the first index is shifted by $|T|$) converging to $H_n^{ch}(A, \{M_s\})$ looks as

$$E^1_{p,q} = (\otimes h(M_s)) \otimes H_{DR}^{-p-q}(\mathcal{E}xp(U)_p, \mathrm{Sym}^p(A[1])). \tag{3.11.24.2}$$

Assume that A is a plain chiral algebra and M_s are plain modules. Then

$$E^1_{p,q} = (\otimes h(M_s)) \otimes H^{DR}_{p+q}(\mathcal{E}xp(U)_p, \Lambda^p_{ext} A). \tag{3.11.24.3}$$

As in 3.11.21 we see that $E^1_{p,q} = 0$ unless $p \geq q \geq 0$. In particular, $H_a^{ch}(A, \{M_s\}) = 0$ for $a < 0$.

Let us compute $\langle h(M_s)\rangle := H_0^{ch}(A, \{M_s\})$. The Lie algebra $h(U, A)$ acts on each $h(M_s)$, so $\otimes h(M_s)$ is an $h(U, A)$-module. One has $E^2_{0,0} = H_0^{ch}(A, \{M_s\})$, so, as in 3.11.22(i), we see that the projection $E^1_{0,0} = \otimes h(M_s) \twoheadrightarrow E^2_{0,0}$ identifies $\langle h(M_s)\rangle$ with the coinvariants of the $h(U, A)$-action on $\otimes h(M_s)$:

$$(\otimes h(M_s))_{h(U,A)} \xrightarrow{\sim} \langle h(M_s)\rangle. \tag{3.11.24.4}$$

More generally, let $P \subset A_U$ be any $\mathcal{D}_U$-submodule that generates A_U as a chiral algebra. Then the coinvariants of the actions of $h(U, P)$ and $h(U, A)$ on $\otimes h(M_s)$ coincide,[249] so we have

$$(\otimes h(M_s))_{h(U,P)} \xrightarrow{\sim} \langle h(M_s)\rangle. \tag{3.11.24.5}$$

Example. Assume that A is flat at S and $M_s = j_{s*}j_s^* A/A$, i.e., $h(M_s) = A_s^\ell$ is the vacuum module. Then $C(A, \{A_s^\ell\}) = i_{S*}Ri_S^! C(A)$ (see 3.11.19 for notation). Therefore, by 3.11.20, one has

$$H_a^{ch}(A, \{A_s^\ell\}) = H_a^{ch}(A). \tag{3.11.24.6}$$

[249] Proof: the class of collections $\{M_s\}$ for which our statement is true is closed under quotients. As follows from 3.11.22(ii) it contains any $\{M_s\}$ such that every $h(M_s)$ is isomorphic to a direct sum of $A_s^{a\,s}$-modules $A_{\xi\,s}$ where $A_\xi \in \Xi_s^{a\,s}$. Since every $A_s^{a\,s}$-module is a quotient of one of these modules we are done.

More generally, if $S' \subset X$ is another finite subset such that $S \cap S' = \emptyset$ and A is flat at S' then $C(A, \{h(M_s), A^\ell_{s'}\}_{S \sqcup S'}) = i_{S'*} Ri^!_{S'} C(A, \{M_s\})$. By the variant of 3.11.20 for $C(A, \{M_s\})$ (see the end of 3.11.23) we have a canonical isomorphism (we use notation of (3.11.24.1))

$$H^{ch}_a(X \smallsetminus S, A, \{M_s\}) = H^{ch}_a(X \smallsetminus (S \sqcup S'), A, \{M_s, A^\ell_{s'}\}). \tag{3.11.24.7}$$

3.11.25. Let L be a DG Lie* algebra on X which we assume to be flat as an $\mathcal{O}_X$-module. Let $S \subset X$ be a finite (possibly empty) subset, $U = U_S := X \smallsetminus S$, M_s a DG chiral L-module (see 3.7.16–3.7.18) supported at $s \in S$.

We are going to compute the chiral homology $R\Gamma^{ch}(U(L), \{M_s\})$. It appears that they coincide with the homology of the DG Lie algebra $R\Gamma_{DR}(U, L)$ with coefficients in $\otimes h(M_s)$.

To make sense to this statement we have to explain first why $R\Gamma_{DR}(U, L)$ is a DG Lie algebra. Precisely, we want to define a canonical object in the homotopy category of DG Lie algebras (see e.g. [H]) which identifies with $R\Gamma_{DR}(U, L)$ as an object of plain derived category of k-modules. Of course[250] $R\Gamma_{DR}(U, L)$ is a Lie algebra in the tensor category $D(k)$, but this does not suffice for our purpose.

The nest point we want to explain is how to write down the Chevalley complex of $R\Gamma_{DR}(U, L)$. The problem is that the construction of 2.2.19 yields directly not a Lie DG algebra, but a homotopy one (there was a twist by a contractible operad $\mathcal{T}$). Of course, the corresponding objects are the same as Lie DG algebras in the homotopy category, but we want to write down the Chevalley complex explicitly.

Let us return to 2.2.19(iv). According to (2.2.19.1), (2.2.19.2) $DR^{\mathcal{S}}(L)$ is a Lie DG algebra in the pseudo-tensor DG category $\mathcal{C}(\mathcal{B}, C(X))$. We say that an object $F \in \mathcal{C}(\mathcal{B}, C(X))$ is a *quasicomplex on* X if for every simplex $\tilde{n} \in \mathcal{B}_l$ and a structure "simplicial map" $\partial : \mathcal{B}_l \to \mathcal{B}_m$ the corresponding morphism of complexes $F_{\tilde{n}} \to F_{\partial \tilde{n}}$ is a quasiisomorphism.

is a Lie algebra in the derived category of complexes, and one has yet to explain how to lift it to an object of the derived category of DG Lie algebras. Instead of doing this we formulate our theorem without mentioning explicitly this DG Lie algebra structure. The

[250] Since $R\Gamma_{DR}$ is a pseudo-tensor functor $D(\mathcal{M}(X))^* \to D(k)$.

homology of the graded Lie algebra $H^{\cdot}_{DR}(U,L)$ appear then on the level of the spectral sequence (see 3.11.26).

Let us define a $\mathcal{D}$-subcomplex[251]

$$C(L,\{M_s\}) \hookrightarrow C(U(L),\{M_s\}). \tag{3.11.23.1}$$

If $S=\emptyset$ then $C(L)=C(\omega)\oplus \bar{C}(L)$. Here the embedding $C(\omega)\hookrightarrow C(U(L))$ is given by $1_{U(L)}$, and the $\mathcal{D}$-subcomplex $\bar{C}(L)\subset C(U(L))$ is defined as follows. Each $\bar{C}(L)^{\cdot}_{X^I}$ is a $Q(I)$-graded submodule of $C(U(L))^{\cdot}$ (see 3.5.11); for $T\in Q(I)$ its T-component is $\Delta_*^{(I/T)}(L[1])^{\boxtimes T}\subset \Delta_*^{(I/T)}j_*^{(T)}j^{(T)*}(U(L)[1])^{\boxtimes T}$. As follows from 3.5.11 the "interesting" components of the differential on $\bar{C}(L)$ are given by the $*$ bracket on L.

If $S\neq\emptyset$ then $C(L,\{M_s\})^{\cdot}_{X^I}$ is a $Q(I)$-graded submodule of $C(U(L),\{M_s\})^{\cdot}_{X^I}$; for $T\in Q(I)$ its T-component is $\Delta_*^{(I/T)}(\underset{S\hookrightarrow T}{\oplus}(\boxtimes M_s)\boxtimes (j_{U*}j_U^*L[1])^{\boxtimes T\smallsetminus S})$. We see that the "interesting" components of the differential are given by the $*$ bracket on L and the chiral action of L on M_s.

Our $\mathcal{D}$-complex $C(L,\{M_s\})$ carries an increasing filtration $C(L,\{M_s\})_{\cdot}$ where the subcomplex $C(L,\{M_s\})_{nX^I}$ is the sum of T-components for $|T|\leq n+|S|$. The complex $C(U(L),\{M_s\})$ also carries an increasing filtration defined by the canonical filtration on $U(L)$ (and the trivial filtration on M_s). The embedding (3.11.23.1) is compatible with the filtrations.

Passing to de Rham cohomology we get a morphism of filtered complexes

$$R\Gamma_{DR}(X^S,C(L,\{M_s\}))\to R\Gamma^{ch}(X,U(L),\{M_s\}). \tag{3.11.23.2}$$

3.11.24 Theorem. This is a filtered quasiisomorphism.

For a proof see 3.11.27.

3.11.25 Corollary. There is a canonical identification

$$\mathrm{gr}_n R\Gamma^{ch}(X,U(L),\{M_s\})\xrightarrow{\sim}(\otimes h(M_s))\otimes \mathrm{Sym}^n(R\Gamma(U,L)[1]). \tag{3.11.25.1}$$

[251] Unlike $C(U(L),\{M_s\})$ the complex $C(L,\{M_s\})$ on X^S is *not* admissible.

The "extension" morphism $\mathrm{gr}_n R\Gamma^{ch}(X, U(L), \{M_s\}) \to \mathrm{gr}_{n-1} R\Gamma^{ch}(X, U(L), \{M_s\})[1]$ is the Chevalley differential $(\otimes h(M_s)) \otimes \Lambda^n R\Gamma(U, L) \to (\otimes h(M_s)) \otimes \Lambda^{n-1} R\Gamma(U, L)$ defined by the Lie bracket on $R\Gamma(U, L)$ and the action of $R\Gamma(U, L)$ on $\otimes h(M_s)$.

Proof of the corollary. As follows from the definition of $C(L, \{M_s\})$ and the sublemma in 3.11.4 the canonical morphism of complexes $(\otimes h(M_s)) \otimes \mathrm{Sym}^n(R\Gamma(U, L)[1]) = R\Gamma_{DR}(X^{n+|S|}, C(L, \{M_s\}))_{\Sigma_{n+|S|}} \to R\Gamma_{DR}(X^S, \mathrm{gr}_n C(L, \{M_s\}))$ is a quasiisomorphism. Now use 3.11.24 to get (3.11.25.1). The second statement is left to the reader. □

3.11.26. The canonical filtration on $U(L)$ yields a spectral sequence converging to the chiral homology $H_n^{ch}(U(L), \{M_s\})$. The above corollary computes it. Namely, consider the graded Lie algebra $H^{\cdot}_{DR}(U, L)$. It acts on $H^{\cdot}(\otimes h(M_s))$. Then the second term of our spectral sequence equals to the Lie algebra homology of $H^{\cdot}_{DR}(U, L)$ with coefficients in $H^{\cdot}(\otimes h(M_s))$.

Important particular cases: if $H^{\neq 0}_{DR}(U, L) = 0$ and M_s are plain chiral L-modules then

$$H_n^{ch}(U(L), \{M_s\}) = H_n(H^0_{DR}(U, L), \otimes h(M_s)). \tag{3.11.25.2}$$

If $S = \emptyset$ and $H^{\neq 1}_{DR}(X, L) = 0$ then

$$H_0^{ch}(X, U(L)) = \mathrm{Sym} H^1_{DR}(X, L), \quad H^{ch}_{\neq 0}(X, U(L)) = 0. \tag{3.11.25.3}$$

3.11.27. *Proof of* 3.11.24. For simplicity of notation we assume that $S = \emptyset$. The similar argument in case $S \neq \emptyset$ is left to the reader.

(i) Denote by L^{ab} the $\mathcal{D}_X$-module L considered as a Lie* algebra with trivial bracket. According to 3.7.14 we have $\mathrm{gr}U(L) = \mathrm{Sym}L = U(L^{ab})$. It is clear that $\mathrm{gr}C(U(L)) = C(\mathrm{Sym}L)$.

Therefore our theorem will follow if we prove that (3.11.23.2) is a quasiisomorphism in case when the $*$ bracket on L is trivial.[252]

(ii) From now on we assume this condition. Then the action of homotheties on L yields $\mathbb{Z}_{\geq 0}$-gradings on our complexes which splits the filtrations. We denote the graded components by $C(L)^{(n)} \subset C(\mathrm{Sym}L)^{(n)}$.

[252] In case $S \neq \emptyset$ you assume in addition that the chiral action of L on M_s is trivial.

We will define on $C(\mathrm{Sym}L)$ a decreasing filtration $F^n = F^nC(\mathrm{Sym}L)$, $n \geq 0$, that satisfies the following properties:

(a) Our filtration is compatible with the grading; the filtration on each graded component $F^0C(\mathrm{Sym}L)^{(n)}$ is finite.

(b) One has $R\Gamma_{DR}(X^{\mathcal{S}}, C(\mathrm{Sym}L)/F^0) = 0$.

(c) One has $C(L)^{(n)} \subset F^n$, and this embedding yields a quasiisomorphism of complexes $\epsilon^n : R\Gamma_{DR}(X^{\mathcal{S}}, C(L)^{(n)}) \xrightarrow{\sim} R\Gamma_{DR}(X^{\mathcal{S}}, \mathrm{gr}^n_F C(\mathrm{Sym}L))$.

This will clearly prove our theorem.

(iii) The filtration F we are going to define is compatible with the $Q(I)$-grading on each $C(\mathrm{Sym}L)^{\cdot}_{X^I}$. So we need to determine for every $T \in Q(I)$ the filtration F^n_T on the T-component of $C(\mathrm{Sym}L)^{\cdot}_{X^I}$. This T-component equals $\Delta_*^{(I/T)} j_*^{(T)} j^{(T)*}((\mathrm{Sym}L)[1])^{\boxtimes T}$ (see 3.5.11).

Set $F^1_T := \Delta_*^{(I/T)} j_*^{(T)} j^{(T)*}((\mathrm{Sym}^{>0}L)[1])^{\boxtimes T}$, $F^0 := C(\omega) \oplus F^1$. To define the next terms of the filtration notice that $j_*^{(T)} j^{(T)*}((\mathrm{Sym}L)[1])^{\boxtimes T} = (j_*^{(T)} j^{(T)*} \omega_X^{\boxtimes T}) \otimes ((\mathrm{Sym}L^\ell)[1])^{\boxtimes T}$. The $\mathcal{D}_T$-module $j_*^{(T)} j^{(T)*} \omega_X^{\boxtimes T}$ carries the canonical filtration W^T_n defined in 3.1.5, 3.1.6. For $n \geq 1$ we set $F^n_T := \Delta_*^{(I/T)}(W^T_{-n} \otimes ((\mathrm{Sym}^{>0}L^\ell)[1])^{\boxtimes T})$.

It is clear that F^n is a filtration by subcomplexes of $C(\mathrm{Sym}L)$.

(iv) It remains to show that our F^n satisfies conditions (a)–(c) of (ii). Condition (a) is obvious.

Let us check (b). The $\mathcal{D}$-complex $[C] := C(\mathrm{Sym}L)/F^0$ carries an obvious filtration $V_1 \subset V_2 \subset ..$ such that the components of $\mathrm{gr}^V_n[C]^{\cdot}_{X^I}$ are $\Delta_*^{(I/S\sqcup T)} j_*^{(S\sqcup T)} j^{(S\sqcup T)*}((\omega[1])^{\boxtimes S} \boxtimes ((\mathrm{Sym}^{>0}L)[1])^{\boxtimes T})$ where $S, T \in \mathcal{S}$, $|T| = n$. It suffices to show that $R\Gamma_{DR}(\mathcal{R}(X), \mathrm{gr}^V_n[C]) = 0$. We have $R\Gamma_{DR}(\mathcal{R}(X), \mathrm{gr}^V_n[C]) = R\Gamma_{DR}(\hat{U}_n, C(\omega) \boxtimes G_n)$ where $\hat{U}_n \subset \mathcal{R}(X) \times \mathcal{R}(X)^o_n$ is the open subset of pairs $([S], [T])$ such that $S \not\subseteq T$ and G_n denotes the $\mathcal{D}$-complex[253] $\mathrm{Sym}^n((\mathrm{Sym}^{>0}L)[1])$ on $\mathcal{R}(X)^o_n$. Let $p_1, p_2 : \hat{U}_n \to \mathcal{R}(X)$, $\mathcal{R}(X)^o_n$ be the projections. A relative version of the argument used in the proof of 3.11.18 shows that $p_{2*}p_1^* C(\omega) = 0$. By projection formula this implies the desired vanishing, and we are done.

[253] Here the first Sym is exterior symmetric product, while the second one is "interior" symmetric algebra.

Let us turn to condition (c). The first statement is obvious, so we have to check that ϵ^n is a quasiisomorphism. For $n = 0$ our statement is obvious. Consider the case $n = 1$. Notice that $\mathrm{gr}^1_F C(\mathrm{Sym}L)_{X^I}$ is supported on the diagonal $X \subset X^I$, so we can rewrite $R\Gamma_{DR}(X^{\mathcal{S}}, \mathrm{gr}^1_F C(\mathrm{Sym}L)$ as the de Rham cohomology of an $\mathcal{S}^\circ$-diagram Φ of $\mathcal{D}$-complexes on X. Using the canonical identification $\mathrm{gr}^W_{-1} j^{(T)}_* j^{(T)*} \omega_X^{\boxtimes T} = (P^{ch}_T(\omega_X))^* \otimes \Delta^{(T)}_* \omega_X = \mathcal{L}ie^*_T \otimes \Delta^{(T)}_* \omega_X$ (see 3.1.4–3.1.6) we see that $\Phi(I) = \underset{T \in Q(I)}{\oplus} \mathcal{L}ie^*_T \otimes ((\mathrm{Sym}^{>0} L)[1])^{\otimes T}$, i.e., $\Phi(I)^\ell = \underset{T \in Q(I)}{\oplus} \mathcal{L}ie^*_T \otimes (\mathrm{Sym}^{>0} L^\ell)^{\otimes T} \lambda_T[|T|]$. Here the differential in Φ comes from the differential in $\mathrm{Sym}^{>0} L^\ell$ and the maps $\mathcal{L}ie^*_T \otimes (\mathrm{Sym}^{>0} L^\ell)^{\otimes T} \to \mathcal{L}ie^*_{T'} \otimes (\mathrm{Sym}^{>0} L^\ell)^{\otimes T'}$ for $T' \subset T$, $|T'| = |T| - 1$, equal to the tensor product of the "cobracket map" $\mathcal{L}ie^*_T \to \mathcal{L}ie^*_{T'}$ and the product map $(\mathrm{Sym}^{>0} L^\ell)^{\otimes T} \to (\mathrm{Sym}^{>0} L^\ell)^{\otimes T'}$.

Denote by $\bar{\Phi}$ the $\mathcal{S}^\circ$-inductive limit of Φ. Consider the canonical map $C(\mathcal{S}^\circ, \Phi) \to \bar{\Phi}$ (we use notation of 3.11.1(iii)). This is a quasiisomorphism[254], so $R\Gamma_{DR}(X^{\mathcal{S}}, \mathrm{gr}^1_F C(\mathrm{Sym}L) = R\Gamma_{DR}(X, \bar{\Phi})$. Notice that $\bar{\Phi}^\ell$ is just the Harrison complex of the commutative algebra[255] $(\mathrm{Sym}^{>0} L^\ell)[1]$. Therefore, by ???, the embedding $L \hookrightarrow \bar{\Phi}$ is a quasiisomorphism, and we are done.

The case $n > 1$ is just exterior tensor product of n copies of the previous situation. □

3.11.27.

§12 Utensils

This section collects some implements needed in §11.

3.12.1 Homotopy direct limits. Below $\mathcal{A}$ is a category which we tacitly assume to be closed under direct sums of sufficiently high cardinality.

(i) Let B be a simplicial set. So for every $n \geq 0$ we have a set B_n and for every monotonous map of intervals $\alpha : [0, .., m] \to [0, .., n]$ we have the corresponding map $\partial_\alpha : B_n \to B_m$ compatible with compositions of α's.

[254] To see this consider an increasing filtration on Φ such that $\Phi_m(I)$ is formed by the summands in $\Phi(I)$ with $|T| \leq m$ and the corresponding filtration on $\bar{\Phi}$. Our morphism is, in fact, a filtred quasiisomorphism; this follows from the sublemma from 3.11.4.

[255] In the tensor category $\mathcal{M}^\ell(X)$.

Denote by $\mathcal{C}(B,\mathcal{A})$ the category of homology type coefficient systems on B with coefficients in $\mathcal{A}$. Thus $F \in \mathcal{C}(B,\mathcal{A})$ is a rule that assigns to every simplex $b \in B_n$ an object $F_b \in \mathcal{A}$ and to every α as above a morphism $\partial_\alpha = \partial_\alpha^F : F_b \to F_{\partial_\alpha(b)}$ compatible with composition of α's.

Denote by $\mathcal{C}_s(\mathcal{A})$ the category of simplicial objects in $\mathcal{A}$. We have a functor

$$C_s : \mathcal{C}(B,\mathcal{A}) \to \mathcal{C}_s(\mathcal{A}) \tag{3.12.1.1}$$

where $C_s(F)_n := \underset{b\in B_n}{\oplus} F_b$ and the structure maps $\partial_\alpha : C_s(F)_n \to C_s(F)_m$ are direct sums of the corresponding morphisms ∂_α^F.

(ii) Let $\mathcal{P}$ be a small category. It yields a simplicial set $B\mathcal{P}$ (see e.g. [QK]); an n-simplex $\tilde{p} \in B\mathcal{P}_n$ is a diagram $(p_0 \to .. \to p_n)$ in $\mathcal{P}$. Consider the category $\mathcal{F}un(\mathcal{P},\mathcal{A})$ of functors $F : \mathcal{P} \to \mathcal{A}$, $p \mapsto F_p$. There is a fully faithful embedding

$$\mathcal{F}un(\mathcal{P},\mathcal{A}) \hookrightarrow \mathcal{C}(B\mathcal{P},\mathcal{A}) \tag{3.12.1.2}$$

which identifies a functor F with the system $F_{(p_0\to..\to p_n)} = F_{p_0}$.

(iii) Assume that $\mathcal{A}$ is a k-category. Let $C(\mathcal{A})$ be the category of complexes in $\mathcal{A}$. By Dold-Puppe one has the fully faithful embedding

$$\mathcal{N}orm : C_s(\mathcal{A}) \hookrightarrow C(\mathcal{A}) \tag{3.12.1.3}$$

which identifies $C_s(\mathcal{A})$ with the full subcategory of complexes sitting in degrees ≤ 0.

(iv) Now assume that $\mathcal{A}$ is a DG (super) k-category. We have the functor tot: $C(\mathcal{A}) \to \mathcal{A}$ which sends a complex $C = (C^n, d)$ to an object $\mathrm{tot}C \in \mathcal{A}$ such that $\mathrm{tot}C = \oplus C^n[-n]$ as a plain object without differential, the structure differential is the sum of d and the structure differentials of $C^n[-n]$. Set

$$\mathrm{Norm} := \mathrm{tot}\mathcal{N}orm : C_s(\mathcal{A}) \to \mathcal{A} \tag{3.12.1.4}$$

For a functor $F : \mathcal{P} \to \mathcal{A}$ set $C(\mathcal{P},F) := \mathrm{Norm}C_s(B\mathcal{P},F)$; this is *the homotopy direct limit* of F.

Notice that the DG structure on $\mathcal{A}$ yields DG structures on all the above categories, and (3.12.1.1)–(3.12.1.4) are DG functors.

Remark. If $\mathcal{A} = C(\mathcal{B})$ and F takes values in $\mathcal{B} \subset \mathcal{A}$ that the complex $C(\mathcal{P}, F)$ lives in degrees ≤ 0 and $H^0C(\mathcal{P}, F) \in \mathcal{B}$ is the plain $\mathcal{P}$-direct limit of F.

3.12.2 Operations I. Below $\mathcal{A}$ is pseudo-tensor category; we assume that direct sums in $\mathcal{A}$ are compatible with operations.[256]

(i) Consider the category of k-modules. This is a tensor category, so both categories of simplicial k-modules and k-complexes are tensor categories. $\mathcal{N}orm$ is *not* a tensor functor. However for any finite family of simplicial k-modules M_i there is a canonical morphism of complexes $c = c_{\{N_i\}} : \otimes \mathcal{N}orm(\mathcal{N}_i) \to \mathcal{N}orm(\otimes N_i)$ uniquely determined by condition that it equals $id_{\otimes N_i}$ if N_i are constant simplicial k-modules.[257] Therefore $\mathcal{N}orm$ is a pseudo-tensor functor.

Now if $\mathcal{C}$ is any pseudo-tensor simplicial k-category we define its *normalization* as a pseudo-tensor DG k-category $\mathcal{N}orm\mathcal{C}$ having same objects as $\mathcal{C}$ and operations $P^{\mathcal{N}orm\mathcal{C}} := \mathcal{N}orm P^{\mathcal{C}}$. The composition of operation is defined using the above c.

Remark. The plain k-categories corresponding to $\mathcal{C}$ and $\mathcal{N}orm\mathcal{C}$ coincide.

(ii) $C_s(\mathcal{A})$ is a simplicial pseudo-tensor catregory. Namely, for $F_i, G \in C_s(\mathcal{A})$ the simplicial set $P_I(\{F_i\}, G)_s$ is defined as follows. Consider $P^a_b := P_I(\{F_{i\,a}\}, G_b)$; they form a cosimplicial-simplicial set P. Our $P_I(\{F_i\}, G)_s$ is the corresponding total simplicial set $\mathcal{T}otP$ (for the definition of $\mathcal{T}ot$ see e.g. [BK]).

Remark. If $\mathcal{A}$ is a tensor category then $C_s(\mathcal{A})$ is a tensor simplicial category, and this tensor structure coincides with the obvious one.

(iii) If $\mathcal{A}$ is a pseudo-tensor k-category then $C(\mathcal{A})$ is a pseudo-tensor DG k-category. The normalization functor extends in the obvious way to a pseudo-tensor DG functor

$$\mathcal{N}orm : \mathcal{N}orm C_s(\mathcal{A}) \to C(\mathcal{A}). \tag{3.12.2.1}$$

[256] i.e., $P_I(\{\underset{\alpha \in A_i}{\oplus} F_{\alpha i}\}, G) = \prod P_I(\{F_{\alpha i}\}, G)$.

[257] Which amounts to the fact that $\mathcal{N}orm(N_i)$ are complexes supported in degree 0.

(iv) Assume that $\mathcal{A}$ is a pseudo-tensor DG (super) k-category. Then the categories in (3.11.2.1) are biDG categories and $\mathcal{N}orm$ is a pseudo-tensor biDG functor. Let us consider them as DG categories (with respect to the total grading and differential).

The functor tot: $C(\mathcal{A}) \to \mathcal{A}$ is a pseudo-tensor DG functor in the obvious way. We get a pseudo-tensor DG functor

$$\text{Norm} := \text{tot}\mathcal{N}orm : C_s(\mathcal{A}) \to \mathcal{A}. \tag{3.12.2.2}$$

3.12.2 Operations II. (i) Let B be a simplicial set equipped with an action of a simplicial operad T. Then $\mathcal{C}(B,\mathcal{A})$ is a pseudo-tensor category: an operation $\varphi \in P_I(\{F_i\}, G)$ is a rule that attaches to every $t \in T_{In}$ and $(b_i) \in B_n^I$ an operation $\varphi_t \in P(\{F_{b_i}\}, G_{t(b_i)})$ such that for every structure map ∂_α (see 3.12.1(i)) one has $\varphi_{\partial_\alpha(t)}(\partial_\alpha^{F_i}) = \partial_\alpha^G \varphi_t$. The composition of operations is the obvious one.

The functor C_s lifts to a simplicial pseudo-tensor functor

$$C_s : \mathcal{C}(B,\mathcal{A}) \times T \to \mathcal{C}_s(\mathcal{A}) \tag{3.12.3.1}$$

In the k-linear setting we can rewrite it as a simplicial pseudo-tensor k-functor

$$C_s : \mathcal{C}(B,\mathcal{A}) \otimes k[T] \to \mathcal{C}_s(\mathcal{A}) \tag{3.12.3.2}$$

Set $\mathcal{T} := \mathcal{N}ormk[T]$; this is a DG operad. If $\mathcal{A}$ is a DG k-category then composing $\mathcal{N}ormC_s$ with tot we get a pseudo-tensor DG functor

$$C : \mathcal{C}(B,\mathcal{A}) \otimes \mathcal{T} \to \mathcal{A}. \tag{3.12.3.3}$$

In particular, for any DG k-operad $\mathcal{B}$ if F is a $\mathcal{B}$ DG algebra is $\mathcal{C}(B,\mathcal{A})$ then $C(F)$ is a DG $\mathcal{B} \otimes \mathcal{T}$ algebra in $\mathcal{A}$.

(ii) Let $\mathcal{P}$ be a small tensor (:= commutative monoidal) category; we denote the operation in $\mathcal{P}$ by Σ. Then for every non-empty finite set I we have a functor $\Sigma_I : \mathcal{P}^I \to \mathcal{P}$ defined up to a canonical isomorphism. All such functors Σ_I form a small category equivalent to the trivial one; let T_I be the corresponding contractible simplicial set (= the simplex with

the set of vertices Σ_I). These T_I's form a simplicial operad T which acts on $B\mathcal{P}$ in the obvious way.

One defines a pseudo-tensor structure on $\mathcal{F}unct(\mathcal{P},\mathcal{A})$ as follows. An operation $\varphi \in P_I(\{F_i\},G)$ is a rule that assigns to every $(p_i) \in \mathcal{P}^I$ an operation $\varphi_{(p_i)} \in P_I(\{F_{ip_i}\},G_{\Sigma p_i})$ such that for every morphism $(f_i):(q_i)\to(p_i)$ in $\mathcal{P}^I$ the operations $\varphi_{(p_i)}(f_i),(\Sigma f_i)\varphi_{(q_i)} \in P_I(\{F_{iq_i}\},G_{\Sigma p_i})$ coincide. It is clear that (3.12.1.2) extends to a fully faithful pseudo-tensor embedding

$$\mathcal{F}unct(\mathcal{P},\mathcal{A}) \hookrightarrow \mathcal{C}(B\mathcal{P},\mathcal{A}). \tag{3.12.3.4}$$

If $\mathcal{A}$ is a pseudo-tensor DG category then this is a pseudo-tensor DG embedding. Composing it with (3.12.3.3) we get a pseudo-tensor DG functor (here $\mathcal{T}=\mathcal{N}ormk[T]$)

$$C(\mathcal{P},\cdot):\mathcal{F}unct(\mathcal{P},\mathcal{A})\otimes\mathcal{T}\to\mathcal{A}. \tag{3.12.3.5}$$

3.12.4.

3.12.5 Dolbeault algebras. In 3.11 we deal with de Rham homology of systems of $\mathcal{D}$-modules on powers of X connected by diagonal embeddings. We will present explicit complexes computing this homology which are compatible with the diagonal Gysin maps and exterior tensor products. These complexes are an algebraic version of the $\bar\partial$-complex. We begin with preliminaries.

Definition. Let X be any scheme. A *Dolbeault $\mathcal{O}_X$-algebra* is a commutative DG $\mathcal{O}_X$-algebra[258] $\mathcal{P}_X$ sitting in degrees ≥ 0 which satisfies the following conditions:

(i) The structure morphism $\mathcal{O}_X \xrightarrow{\alpha} \mathcal{P}_X$ is a quasiisomorphism.

(ii) $\mathcal{P}_X$ is $\mathcal{O}_X$-flat.

(iii) $\mathrm{Spec}\mathcal{P}_X^0$ is an affine scheme.

A Dolbeault $\mathcal{D}_X$-algebra is a commutative DG $\mathcal{D}_X$-algebra which is a Dolbeault $\mathcal{O}_X$-algebra.

Lemma (i) Dolbeault $\mathcal{O}_X$-algebras on a quasiprojective X exist.

[258] As always, quasicoherent as an $\mathcal{O}_X$-module.

(ii) If, in addition, X is smooth then it admits a Dolbeault $\mathcal{D}_X$-algebra.

Proof (i) Let $\pi : Y \to X$ be a Jounalou map, i.e., Y is a torsor over X with respect to an action of some vector bundle such that Y is an affine scheme. Recall that one can construct such Y as follows. Choose an open embedding $X \subset \bar{X}$ such that $\bar{X}$ is a projective variety. Blowing $\bar{X} \smallsetminus X$ up if necessary we can assume that $\bar{X} \smallsetminus X \subset \bar{X}$ is locally defined by one equation. Choose an embedding $\bar{X} \hookrightarrow \mathbb{P}^n$. Let $V \subset \mathbb{P}^{n*} \times \mathbb{P}^n$ be the complement to the incidence divisor. Set $Y := X \underset{\mathbb{P}^n}{\times} V$.

Consider now the relative de Rham complex $DR(Y/X)$. It is clear that $\mathcal{P}_X := \pi_* DR_{Y/X}$ is a Dolbeault $\mathcal{O}_X$-algebra.

(ii) Take $\mathcal{P}_X$ as above. Then the jet algebra $\mathcal{J}\mathcal{P}_X = DR(\mathcal{J}\mathcal{P}_X^0/\mathcal{O}_X)$[259] (see 2.3.2) is a Dolbeault $\mathcal{D}_X$-algebra. Indeed, it obviously satisfies conditions (i), (ii), and (iii) follows since the morphism $\mathrm{Spec}\mathcal{J}\mathcal{P}_X^0 = \mathcal{J}Y \to Y$ is affine. □

One uses Dolbeault algebras to construct canonical resolutions computing the cohomology. In $\mathcal{D}$-module situation the statement looks as follows. Let $\mathcal{Q}_X$ be a Dolbeaut $\mathcal{D}_X$-algebra, M a right $\mathcal{D}_X$-complex.

Lemma. (i) The morphism $\alpha_M = \alpha \otimes id_M : M \to \mathcal{Q} \otimes M$ is a quasiisomorphism.

(ii) One has $R\Gamma_{DR}(X, M) = \Gamma_{DR}(X, \mathcal{Q} \otimes M)$.

Proof (i) If M is a $\mathcal{D}$- (or $\mathcal{O}$-) module then α_M is a quasiisomorphism by (i), (ii).[260] Since $\mathcal{Q}_X$ sits in degrees ≥ 0 this implies that α_M is a quasiisomorphism for every $\mathcal{D}$- (or $\mathcal{O}$-) complex M.

(ii) Let F be a k-sheaf which admits a structure of $\mathcal{O}_X$-quasicoherent $\mathcal{Q}_X^0$-module (we forget about the $\mathcal{D}$-module structure on $\mathcal{Q}_X^0$ here). Then, by condition (iii), one has $H^{>0}(X, F) = 0$. Thus for a complex G of such k-sheaves one has $\Gamma(X, G) = R\Gamma(X, G)$.

Therefore $\Gamma_{DR}(X, \mathcal{Q}_X \otimes M) \xrightarrow{\sim} R\Gamma_{DR}(X, \mathcal{Q}_X \otimes M)$ since $DR(\mathcal{Q}_X \otimes M)$ obviously satisfies this property. By (i) $R\Gamma_{DR}(X, M) \xrightarrow{\sim} R\Gamma_{DR}(X, \mathcal{Q}_X \otimes M)$ and we are done. □

[259] The fact that functors DR and $\mathcal{J}$ commute follows from their universal properties. Precisely, for an $\mathcal{O}_X$-algebra B both $DR(\mathcal{J}B)$ and $\mathcal{J}DR(B)$ represent the same functor $F \mapsto \mathrm{Hom}_{\mathcal{O}_X}(B, F^0)$ on the category of commutative DG $\mathcal{D}_X$-algebras.

[260] Since $\mathcal{O}$ has finite Tor-dimension we can play with infinite complexes without problems.

Remarks. (i) Of course, Dolbeault $\mathcal{D}$-algebras are huge, so the above complexes "computing" $R\Gamma_{DR}(X,M)$ look excessive. A smaller complex doing same job is $\Gamma_{DR}(Y,\pi^*M)$ where $\pi: Y \to X$ is a Jounalou map. However we will actually use Dolbeault $\mathcal{D}$-resolutions for computations of de Rham homology of $\mathcal{D}$-*modules on* $\mathcal{R}(X)$ (see the next subsection), and we do not know how to do this using smaller complexes.

(ii) Let $i: X \hookrightarrow Z$ be a closed embedding of smooth varieties. The for $\mathcal{Q}_X$, M as above one has $\Gamma(Y, DR(i_*(\mathcal{Q}_X \otimes M))) = R\Gamma_{DR}(Z, i_*(\mathcal{Q}_X \otimes M))$. Indeed, terms of the complex $DR(i_*(\mathcal{Q}_X \otimes M))$ admit an increasing bounded below filtration whose successive quotients are $\mathcal{Q}_X^0$-modules; then follow as above.

(iii) According to Remark (i) in 1.4.6 the functor $C\mathcal{M}(X) \to C\mathcal{M}(X, \mathcal{Q}_X)$, $M \mapsto \mathcal{Q}_X \otimes M$, is a compound tensor DG functor (here C denotes the category of complexes or DG modules). The "forgetting of $\mathcal{Q}_X$-action" functor $C\mathcal{M}(X, \mathcal{Q}_X) \to C\mathcal{M}(X)$ is a $*$ pseudo-tensor functor. So $\alpha_M : M \to \mathcal{Q}_X \otimes M$ is a morphism of $*$ pseudo-tensor functors. Thus for $\varphi \in P_I^*(\{M_i\}, N)$ its action on $R\Gamma_{DR}$ coincides with the morphism of complexes $\underset{I}{\boxtimes} \Gamma_{DR}(X, \mathcal{Q}_X \otimes M_i) \to \Gamma_{DR}(X^I, \Delta_*^{(I)}\mathcal{Q}_X \otimes N)$.

3.12.6 Computation of $R\Gamma^{ch}$. We are in the setting of 3.11, so our X is a compact curve. Let $\mathcal{Q}_X$ be a Dolbeault $\mathcal{D}_X$-algebra. According to 3.5.? it yields a commutative factorization DG algebra $\mathcal{Q}$, so we have commutative DG $\mathcal{D}_{X^n}$-algebras $\mathcal{Q}_{X^n}$.

Lemma. Every $\mathcal{Q}_{X^n}$ is a Dolbeault $\mathcal{D}_{X^n}$-algebra.

Proof. Condition (i) holds since it obviously holds on the level of the Cousin complexes. Condition (ii) follows from Remark in 3.5.4. By construction, $\mathcal{Q}_{X^n}$ is a $\mathcal{Q}_X^{\boxtimes n}$-algebra, so one has an affine morphism $\mathrm{Spec}\mathcal{Q}_{X^n} \to (\mathrm{Spec}\mathcal{Q}_X)^n$. The latter is an affine scheme which implies condition (iii). □

Let M be a right $\mathcal{D}$-complex on X^S. We say that M is *convenient* if $\Gamma(X^n, DR(M)) = R\Gamma(X^n, DR(M))$ for every n. A DG chiral algebra A is convenient if such is the $\mathcal{D}$-complex $C(A)$. Thus for convenient M, A one has canonical filtered quasiisomorphisms $R\Gamma_{DR}(X^S, M) = \Gamma_{DR}(X^S, M)$, $R\Gamma^{ch}(X, A) = \Gamma^{ch}(X, A)$.

Corollary. (i) For every right $\mathcal{D}$-complex M on X^S the complex $\mathcal{Q} \otimes M$ (see Remark

in 3.11.6) is convenient.

(ii) For every DG chiral algebra A the chiral algebra $\Omega_X \otimes A$ is convenient.

Proof. (i) is clear, (ii) follows from (i) since $C(\Omega_X \otimes A) = \Omega \otimes C(A)$. □

The canonical morphisms $M \to \Omega \otimes M$, $A \to \Omega_X \otimes A$ are quasiisomorphisms so one has canonical (filtered) quasiisomorphisms

$$R\Gamma_{DR}(X^S, M) = \Gamma_{DR}(X^S, \Omega \otimes M), \quad R\Gamma^{ch}(X, A) = \Gamma^{ch}(X, \Omega_X \otimes A). \tag{3.12.6.1}$$

3.12.7 Batalin-Vilkovisky algebras. For more details and examples see e.g. [Ge]. Below we follow superconventions of 1.1.16 as always dropping the prefix "super".

The Batalin-Vilkovisky operad BV is the odd Poisson k-operad[261] considered as a DG operad. The grading is determined by property that the product $\cdot$ has degree 0 and the odd Poisson bracket $\{\ \ \}$ has degree 1. The differential is determined by property $d(\cdot) = \{\ \ \}$. So for a complex C a BV algebra structure on C is a commutative algebra structure on C considered as a plain graded (super) vector space given by product $\cdot$ together with a Lie bracket $\{\ \ \}$ on $C[-1]$. One demands that the Lie algebra $C[-1]$ acts on C (the adjoint action) by derivations of $\cdot$ and $d(\cdot) = \{\ \ \}$.

The BV operad is acyclic – one has $H^\cdot(\mathrm{BV}_n) = 0$ for $n > 0$. So to get an interesting homotopy theory one has to consider BV as a *filtered* DG operad.

Namely, BV carries a canonical increasing filtration (the stupid one) $BV^{\geq -n}$. A *filtered* BV algebra is a complex C equipped with a BV algebra structure and an increasing filtration which is compatible with the BV algebra structure (i.e., the products $\mathrm{BV}_n \otimes C^{\otimes n} \to C$ are compatible with the filtrations). This amounts to the property that the product $\cdot$ is compatible with filtrations and the induced product on $\mathrm{gr}C$ is compatible with the differential. Therefore $\mathrm{gr}C$ is an odd Poisson DG algebra.[262]

From now on we assume that our filtered BV algebras C satisfy conditions

$$C_{-1} = 0, \quad \cup C_n = C. \tag{3.12.7.1}$$

[261] alias braid, alias Gerstenhaber operad.

[262] The odd Poisson bracket $\{\ \ \} : \mathrm{gr}_i C[-1] \otimes \mathrm{gr}_j C[-1] \to \mathrm{gr}_{i+j-1} C[-1]$ comes from the bracket $\{\ \ \} : C_i[-1] \otimes C_j[-1] \to C_{i+j-1}[-1]$.

The category of such filtered BV algebras is denoted by $\mathcal{BV}$.

The BV operad is augmented in the obvious way. So we have the notion of *unital* BV algebra. Explicitely, a BV algebra C is unital if it has a unit $1 \in C^0$ with respect to $\cdot$ such that $d(1) = 0$ (then 1 lies in the $\{\ \ \}$-center of C). In the filtered setting we assume that $1 \in C_0$. The category of unital BV algebras is denoted by $\mathcal{BV}u$.

The DG operad BV has a canonical coproduct $\delta : \mathrm{BV} \to \mathrm{BV} \otimes \mathrm{BV}$, $\delta(\cdot) = \cdot \otimes \cdot$, so we know what is tensor product of BV algebras.[263] Tensor product of BV compatible filtrations is BV compatible, so we know what is tensor product of filtered BV algebras. Tensor product of unital algebras is obviously unital.

Following [H] one can show that the category $\mathcal{BV}$ is a closed model category in the sense of Quillen: the weak equivalences are filtered quasiisomorphisms, the fibrations are those morphisms f that $\mathrm{gr}f$ is surjective. Therefore one can localize it by filtered quasiisomorphisms in a nice way; the result is the homotopy category $\mathcal{H}o\mathcal{BV}$ of BV algebras. In the unital setting we get the homotopy category $\mathcal{H}o\mathcal{BV}u$.

Remark. The notion of BV algebra makes sense in any DG pseudo-tensor category, that of filtered BV algebra in a filtered DG pseudo-tensor category. Unital algebras make sense in the augmented setting (see 1.2.8).

3.12.8 Enveloping BV algebras. For a Lie DG algebra L its Chevalley homology complex $C(L)$ is a filtered unital BV algebra: as a plain graded algebra it equals $\mathrm{Sym}(L[1])$, the odd Poisson bracket is determined by condition that on $L = \mathrm{Sym}^1(L[1])[-1] \subset C[-1]$ it coincides with the Lie bracket, the filtration is $\mathrm{Sym}^{\leq i}(L[1])$.

The above example can be generalized as follows. Let R be a commutative unital DG algebra. For a Lie DG R-algebroid[264] $\mathcal{L}$ its BV *extension* is an extension of complexes $0 \to R[-1] \xrightarrow{i} \mathcal{L}^\flat \xrightarrow{\pi} \mathcal{L} \to 0$ together with a graded Lie R-algebroid structure on $\mathcal{L}^\flat$ (with differential forgotten) such that

[263] This is the usual tensor product of odd Poisson algebras with the obvious differential.
[264] See 2.8.1.

(a) π is a morphism of graded Lie algebroids, i is a morphism of R-modules, and $k[-1]1^\flat$ belongs to the center of $\mathcal{L}^\flat$; here $1^\flat := i(1) \in \mathcal{L}^\flat[1]$.

(b) The morphism[265] $\mathcal{L}^\flat \otimes R = R \otimes \mathcal{L}^\flat \xrightarrow{d(\cdot)} \mathcal{L}^\flat[1]$ equals $-i$ composed with the structure action of $\mathcal{L}^\flat$ on R.

We call such $(\mathcal{L}, \mathcal{L}^\flat)$ a *BV Lie R-algebroid*, and write it simply as $\mathcal{L}^\flat$. Pairs $(R, \mathcal{L}^\flat)$ form a category in the obvious way.

Now if C is a unital filtered BV algebra then $R_C := C_0 = \mathrm{gr}_0 C$ is a commutative unital DG algebra, $\mathcal{L} := \mathrm{gr}_1 C[-1]$ is a Lie DG R_C-algebroid, and $C_1[-1]$ is a BV extension of $\mathcal{L}$. We have defined a functor from the category of unital BV algebras to that of pairs $(R, \mathcal{L}^\flat)$. It admits a left adjoint functor $(R, \mathcal{L}^\flat) \mapsto C(R, \mathcal{L}^\flat)$. As a plain commutative graded algebra $C(R, \mathcal{L}^\flat)$ equals $\mathrm{Sym}^\flat(\mathcal{L}[1]) :=$ the quotient of $\mathrm{Sym}_R(\mathcal{L}^\flat[1])$ modulo the relation $1^\flat = 1$, the filtration subspaces $C(R, \mathcal{L}^\flat)_a$ are images of $\mathrm{Sym}^{\leq a}(\mathcal{L}^\flat)$, and the odd Poisson bracket and differential are uniquely determined by condition that $\mathcal{L}^\flat \to C(R, \mathcal{L}^\flat)_1[-1]$ is a morphism of Lie DG algebras. If $\mathcal{L}$ is R-flat then $\mathrm{gr}C(R, \mathcal{L}^\flat)$ equals $\mathrm{Sym}(\mathcal{L}[1])$ as an odd Poisson DG algebra.

3.12.9 Homotopy BV algebras. Let $\mathcal{T}$ be a k-operad equipped with a morphism $\mathcal{T} \to \mathcal{C}om$ such that every $\mathcal{T}_I \to \mathcal{C}om_I = k$ is a quasiisomorphism. Then $\mathrm{BV} \otimes \mathcal{T}$ is a filtered DG operad equipped with a morphism of $\mathrm{BV} \otimes \mathcal{T} \to \mathrm{BV}$ which is a filtered quasiisomorphism.

Such objects are called *homotopy BV operads*, and the ones we need are all come from some $\mathcal{T}$ by the above construction.

is a filtered DG operad $\mathrm{BV}^\sim$ equipped with a morphism $\mathrm{BV}^\sim \to \mathrm{BV}$ which is a quasiisomorphism. We assume that the 0th term of the filtration equals $\mathrm{BV}^\sim$ and on every complex of n-operations $\mathrm{BV}^\sim_n$ the filtration is finite. As above the $\mathrm{BV}^\sim$ filtered DG algebras C we consider are assumed to satisfy the condition $C_{-1} = 0$; such algebras form the category $\mathcal{BV}^\sim$. As above, this is is a closed model category in the sense of Quillen (same definition of weak equivalences and fibrations). As in [H] one can show that the obvious functor $\mathcal{BV} \to \mathcal{BV}^\sim$ yields an equivalence of the homotopy categories $\mathcal{H}o\mathcal{BV} \to \mathcal{H}o\mathcal{BV}^\sim$.

[265] Here $\cdot$ is the R-action on $\mathcal{L}^\flat$.

References

[A] F. Akman, *A characterization of the differential in semi-infinite cohomology*, J. Algebra **162, no. 1** (1993), 194–209.

[BB] A. Beilinson and J. Bernstein, *A proof of Jantzen conjectures*, AMS, Advances in Soviet Mathematics **16, part 1** (1993), 1–50.

[BD] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's Hamiltonians and Hecke eigensheaves.*, Preprint (1999).

[BS] A. Beilinson and V. Schechtman, *Determinant bundles and Virasoro algebras*, Commun. Math. Phys. **118** (1988), 651–701.

[BPZ] A. Belavin, A. Polyakov, and A. Zamolodchikov, *Infinite conformal symmetries in two-dimensional quantum field theory*, Nucl. Phys. **B241** (1984), 333–380.

[B] A. Borel, *Aglebraic D-modules*, Academic Press, 1987.

[Bor] R. Borcherds, *Vertex algebras, Kac-Moody algebras and the monster*, Proc. Natl. Acad. Sci. USA **83** (1986), 3068–3071.

[BU] K. Borsuk and S.Ulam, *On symmetric products of topological spaces*, Bull.Amer.Math.Soc. **37** (1931), 875–882.

[Bo] R. Bott, *On the third symmetric potency of* S^1, Fund.Math. **39** (1952), 264–268.

[BK] A. K. Bousfield and D. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math. **304** (1987), Springer-Verlag.

[C] F. R. Cohen, *The homology of* C_{n+1}*-spaces,* $n \geq 0$, Lecture Notes in Math. **533** (1976), Springer-Verlag, 207–351.

[CN] D. W. Curtis and N. T. Nhu, *Hyperspaces of finite subsets which are homeomorphic to* $\aleph_0$*-dimensional linear metric spaces*, Topology and its Applications **19** (1985), 251–260.

[DJKM] E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, *Transformation group for soliton equations I*, Proc. Japan Acad. **57A** (1981), 342–347.

[D] P. Deligne, *Categories Tanakiennes*, The Grothendieck Festschrift, vol. 2, Progress in Math., vol. 87, Birkhauser, Boston, 1990, pp. 111–195.

[DM] P. Deligne, J. Morgan, *Notes on Supersymmetry (following Joseph Bernstein)*, Quantum fields and Strings: a course for mathematicians, vol. 1, AMS-IAS, 1999, pp. 41–97.

[DL] C. Dong and J. Lepowski, *Generalized vertex algebras and relative vertex operators*, Progress in Math., vol. 112, Birkhauser, Boston, 1993.

[DG1] I. Dorfman and I. Gelfand, *Hamiltonian operators and related algebraic structures*, Funct. Anal. Appl. **13, no. 4** (1979), 13–30 (Russian), 248–262 (English).

[DG2] I. Dorfman and I. Gelfand, *Hamiltonian operators and infinite-dimensional Lie aglebras*, Funct. Anal. Appl. **15, no. 3** (1981), 23–40 (Russian), 173–187 (English).

[F] B. Feigin, *The semi-infinite cohomology of Kac-Moody and Virasoro algebras*, Russian Math Surveys **39** (1984), 155–156.

[FBZ] E. Frenkel and D. Ben-Zvi, *Vertex Algebras and Algebraic Curves* (2000).

[FLM] I. Frenkel, J. Lepowski, and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., vol. 134, Academic Press, Boston, 1988.

[FGZ] I. Frenkel, H. Garland, and G. Zuckerman, *Semi-infinite cohomology and string theory*, Proc. Nat. Acad. Sci. U.S.A. **83, no. 22** (1986), 8442–8446.

[FHL] I. Frenkel, Y. Huang, and J. Lepowski, *On Axiomatic Approaches to Vertex Operator Algebras and Modules*, Memoirs of the AMS, 494, vol. 104, 1993.

[Ga] D. Gaitsgory, *Notes on 2d Conformal Field Theory and String Theory*, Quantum fields and Strings: a course for mathematicians, vol. 2, AMS-IAS, 1999, pp. 1017–1090.

[Ge] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Commun. Math. Phys. **159** (1994), 265–285.

[GK] V. Ginzburg and M. Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1994), 203–272.

[GMS] V. Gorbounov, F. Malikov, V. Schechtman, *Gerbes of chiral differential operators II*, math.AG 0003170.

[G] M. Gromov, *Partial Differential Relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, 1986.

[Gr] A. Grothendieck, *Catégories cofibrées additives et complexe cotangent relatif*, Lecture Notes in Math. **79** (1968), Springer-Verlag.

[H] V. Hinich, *Homological algebra of homotopy algebras*, Communications in Algebra **25(10)** (1997), 3291–3323.

[Il] L. Illusie, *Complexe cotangent et déformations.I*, Lecture Notes in Math. **239** (1971), Springer-Verlag.

[K] V. Kac, *Vertex Algebras for Beginners*, University Lecture Series, vol. 10, Amer. Math. Society, 1997.

[KP] V. Kac and D. Peterson, *Spin and wedge representations of infinite-dimensional Lie algebras and groups*, Proc. Nat. Ac. Sci. USA **78** (1981), 3308–3312.

[Ka] I. Kaplansky, *Projective modules*, Annals of Math. **68** (1958), 372–377. Reprinted in Selected Papers by I.Kaplansky (Springer-Verlag, 1995).

[Kap] M. Kapranov, *Operads and algebraic geometry*, Documenta Mathematica, ICM 1998, vol II, 277–286.

[KM] N. Katz and B. Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton Univ. Press, Princeton, 1985.

[Ku] B. Kupershmidt, *Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalisms*, Lecture Notes in Math. **775** (1980), Springer-Verlag, 162–218.

[KuM] B. Kupershmidt and Yu. Manin, *Equations of long waves with a free surface.I. Conservation laws and solutions*, Funct. Anal. Appl. **11, no. 3** (1977), 31–42 (Russian).

[La] J. Lambek, Deductive systems and categories, Lecture Notes in Math. **86** (1969), Springer-Verlag, 76–122.

[Lef] S. Lefschetz, *Algebraic Topology*, AMS Colloquium Publications, vol. 27, Amer. Math. Society, 1942.

[LZ] B. Lian and G. Zuckerman, *Commutative Quantum Operator Algebras*, preprint (1995), Yale University.

[Li] F.E.J.Linton,, *The multilinear Yoneda lemmas: Toccata, fugue, and fantasia on themes by Eilenberg-Kelly and Yoneda*, Lecture Notes in Math. **195** (1971), Springer-Verlag, 209–229.

[L] J.-L. Loday, *La renaissance des opérades*, Sém. Bourbaki 1994/95, exp. 792, Astérisque, vol. 237, 1996, pp. 47–74.

[M] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Math. Soc. Lecture Note Ser., vol. 124, Cambridge Univ. Press, Cambridge, 1987.

[MSV] F. Malikov, V. Schechtman, A. Vaintrob, *Chiral de Rham complex*, Commun. Math. Phys. **204** (1999), 439–473.

[Ma] Yu. Manin, *Algebraic aspects of non-linear differential equations*, Itogi Nauki, Sovremennye Problemy Matematiki **11** (1978), 5–152 (Russian); English translation in Journal Sov. Math.,vol 11 (1979), 1–122..

[May] P. May, *Geometry of iterated loop spaces*, Lecture Notes in Math. **271** (1972), Springer-Verlag.

[P] M. Primc, *Vertex algebras generated by Lie algebras*, math. QA 9901095.

[QK] D. Quillen, *Higher algebraic K-theory I*, Lecture Notes in Math. **341** (1973), Springer-Verlag, 85–147.

[QR] D. Quillen, *Rational homotopy theory*, Ann. Math. **90** (1968), 205–295.

[R] Z. Ran, *Derivatives of moduli*, Internat. Math. Res. Notices, **no. 4** (1993), 93–106.

[RG] M. Raynaud and L. Gruson, *Critères de platitude et de projectivité*, Inventiones Math. **13** (1971), 1–89.

[Rin] G. S. Rinehart, *Differential forms on general Lie algebras*, Trans.Amer.Math.Soc. **108** (1963), 195–222.

[S] M. Saito, *Induced D-modules and differential complexes*, Bull. Soc. Math. Fr. **117** (1989), 361–387.

[SW] G. Segal and G. Wilson, *Loop groups and equations of KdV type*, Publ. IHES **61** (1985), 5–65.

[Sp] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), 121–154.

[T] J. Tate, *Residues of differentials on curves*, Ann. Sci. ENS **4 ser., t.1** (1968), 149–159.